DYNAMIC STABILITY OF PIPES CONVEYING FLUID BY FINITE ELEMENT METHOD

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NOMENCLATURE

A_f	Area of the passage through which the fluid passes
$[c]^{(e)}$	Damping matrix of the element
$[C]$	Assembled damping matrix after applying boundary conditions
$[C]_{BB}$	Assembled damping matrix before applying boundary conditions
$[D]$	Dynamical matrix
E	Modulus of elasticity of the pipe material
f	Number of degrees of freedom assigned to a finite element
I	Moment of inertia of the pipe
$Im(\omega)$	Imaginary part of the eigenvalue
k_{d1}, k_{d2}	Displacement spring constants at left and right ends
k_{t1}, k_{t2}	Rotational spring constants at left and right ends
$[k]^{(e)}$	Stiffness matrix of the element
$[K]$	Assembled stiffness matrix after applying boundary conditions
$[K]_{BB}$	Assembled stiffness matrix before applying boundary conditions
$[\bar{K}]$	Stiffness matrix for the dynamical matrix
l	Length of the finite element
L	Length of the pipe
m	Number of nodes

m_f	Mass of the fluid per unit length of the pipe
m_p	Mass of the pipe per unit length
$[m]^{(e)}$	Mass matrix of the element
$[M]$	Assembled mass matrix after applying boundary conditions
$[M]_{BB}$	Assembled mass matrix before applying boundary conditions
$[\bar{M}]$	Mass matrix for the dynamical matrix
$Ni(x)$	Interpolation functions
$\lfloor N \rfloor$	Interpolation matrix
$\lfloor N' \rfloor$	First derivative of N with respect to x
$\lfloor N'' \rfloor$	Second derivative of N with respect to x
P	Pressure of the fluid
$Re(\omega)$	Real part of the eigenvalue
t	Time
T	Externally applied tension to the pipe
u	Dimensionless fluid flow velocity
V	Velocity of the fluid passing through the pipe
x	Axial co-ordinate of the pipe
$\{y(t)\}$	$\begin{Bmatrix} \{\dot{z}(t)\} \\ \{z(t)\} \end{Bmatrix}$
$\{Y(t)\}$	$\{0\}$
z	Transverse displacement of the pipe
$z_i(t)$	Typical element of the nodal displacement matrix $\{z\}^{ne}$

z_j, z_k	Transverse displacements at nodes j and k, respectively
$\{z\}^n$	Nodal displacement matrix of the pipe
$\{\dot{z}\}^n$	Nodal velocity matrix of the pipe
$\{\ddot{z}\}^n$	Nodal acceleration matrix of the pipe
$\{z\}^{ne}$	Nodal displacement matrix of the finite element
$\{\dot{z}\}^{ne}$	Nodal velocity matrix of the finite element
$\{\ddot{z}\}^{ne}$	Nodal acceleration matrix of the finite element
α_d	Dimensionless displacement spring factor
α_t	Dimensionless rotational spring factor
β	Dimensionless mass
θ_j, θ_k	Slopes $\frac{z}{x}$, at the nodes j and k, respectively
ξ_1	Length coordinate $= 1 - \frac{x}{l}$
ξ_2	Length coordinate $= \frac{x}{l}$
$\{\ \}$	Column matrix
$\lfloor\ \rfloor$	Row matrix
$[\ ]$	Square matrix
$\cdot$	First derivature with respect to time
$\cdot\cdot$	Second derivature with respect to time
$'$	First derivature with respect to x
$''$	Second derivature with respect to x

SYNOPSIS

Dynamic stability of pipes conveying fluid has been studied using finite element analysis. The element matrices are obtained using Galerkin's method. It is shown that this method gives a systematic and straight forward way to account for natural boundary conditions unlike other methods. The results obtained through finite element analysis agree very well with those obtained by using classical methods. The stability of various pipe configurations, for example pipes with various types of elastic supports at both ends and fixed-pinned configuration, have been studied by this method.

A very general computer programme has been developed to solve these problems. An important feature of the programme is its flexibility. Solution for different boundary conditions can be obtained simply by changing the input cards. Also, generalization for non-uniform cross sections and/or multispan pipes with non-periodic and non-identical supports can be easily incorporated.

CHAPTER 1

INTRODUCTION

1.1 Dynamic Stability of Pipes Conveying Fluid by FEM

Lateral vibration of pipes conveying fluid has been studied extensively in the last two decades. The dynamic behavior of these pipes subjected to transverse loading is of great importance in designing systems such as oil pipe lines, heat exchanger tubes, feed lines to rocket motors and water turbines, etc. These problems belong to a branch of engineering mechanics which has come to be known as "flow-induced vibrations". The first book on this subject has been published recently, Blevins [3].

Fluid flowing through a system is a source of energy that can induce vibrations and lead to instabilities. As the flow velocity increases, a certain value is reached at which the pipe looses stability by buckling and/or flutter, and the velocity at which this occurs first is called critical velocity.

The stability of fluid conveying pipes is of practical importance because the natural frequency of a

pipe generally decreases with increasing velocity of fluid flow. The pipe may become susceptible to resonance if its natural frequency falls below certain limits.

The analysis of such problems when done by classical methods is quite involved even for uniform pipes with simple boundary conditions. Also for every case of the boundary condition a fresh characteristic equation has to be solved.

For complex pipe line problems with non-uniform oross sections or multispan pipes with non-periodic, non-identical supports, finite-element analysis is found to be very convenient. Moreover, all types of boundary conditions can also be accommodated very easily.

1.2 Review of Previous Work:

A comprehensive review of the literature with extensive references is given in Paidoussis and Issid [17] and Chen [5].

Historically, in the early 1950's, observation of bending vibrations of the Trans - Arabian pipe-line prompted an investigation by Ashley and Haniland [1]. They studied the vibration of a simply supported pipe and suspected some agency which damps the pipe line motion and tends to destroy its simple harmonic nature. The same

problem was studied independently by Housner [10] using a different approach. He found that at low fluid velocities the effect upon the vibrations of the pipe was negligible but at a certain high velocity the pipe buckled, like a column subjected to axial loading. A general study was made by Niordson [15] who used classical shell theory to deduce the equation of motion for axial, peripheral, and radial deformation of the shell, which led to the same conclusions regarding the stability of a pipe with simply supported ends.

Long [13] took the case of cantilever pipes conveying fluid. He adapted an iterative procedure using a power series for the mode shape, which was applicable to relatively small flow velocities and confirmed experimentally that the forced motion of cantilever pipes were damped by internal flow in the range of flow velocities considered.

Benjamin [2] dealing with the general dynamical problem of articulated pipe system conveying fluid was the first to anticipate the phenomenon of unstable oscillations when such system possesses one free end. He produced a complete theory, supported by experiments for articulated pipe system. He showed that when the pipe is vertical both oscillation and buckling instabilities are possible

in general, whereas when the motion is confined to a horizontal plane, that is when gravity is insignificant buckling can not occur. Later Gregory and Paidoussis [8] showed theoretically and experimentally that at sufficiently high flow velocities cantilever pipes are subjected to oscillatory instability only. Next Paidoussis [17] included the gravity forces and found that buckling instability is not possible in cases of hanging cantilevers only a flutter type instability is possible. This paradox was clarified by Paidoussis and Deksins [18] .

Naguleswaran and Williams [16] studied the effect of internal pressure of the fluid theoretically and experimentally. In the case of pipes supported at both ends the effect of internal pressure was found to be similar to that of flow velocity; that is, the pipe can buckle even at very low flow velocity by the action of sufficiently high internal pressure.

More recently in 1971 Chen [4] studied the stability of a pipe conveying fluid, with the upstream end of the pipe fixed and the downstream end supported by a displacement spring, so that the boundary conditions were intermediate between clamped - free and clamped - pinned; accordingly both buckling and flutter were possible depending on the spring constant.

In a recent paper, Paidoussis and Issid [19] considered the dynamics and stability of flexible pipes containing flowing fluid, where the flow velocity was either entirely constant, or had a small harmonic component superposed. It was shown that conservative systems were subjected not only to buckling but also to oscillatory instability (flutter) at higher flow velocities. The dynamics of a fluid-conveying pipe, clamped at the upstream end and supported by a displacement and a rotational spring at the downstream end was studied by Lin and Chen [12]. It was shown that the pipe may lose its stability by buckling, flutter, or both, depending on the magnitudes of the displacement as well as rotational spring constants.

1.3 Present Work, Objective and Scope:

The aim of the present work is to develop the finite element method for solving the problem of dynamics of pipes conveying fluid.

In the second chapter the equations of motion (in the matrix form) are obtained by finite element analysis using Galerkin's method. The stiffness, damping and mass matrices for a typical element are obtained. Here it is seen that Galerkin's method of finite element analysis

gives very distinctly a way to account for natural boundary conditions. It is noted here that the stiffness and the mass matrices are symmetric but the damping matrix is non-symmetric. These matrices are arranged to form the standard dynamical matrix, Meirovitch [14].

Using a library subroutine the complex eigenvalues of the dynamical matrix are obtained for various boundary conditions of the pipe problem. These results are given in the third chapter and discussed. The mechanism of instability is also discussed at the end of this chapter.

In the fourth chapter the conclusions are reported. Lastly the listing of the detailed Computer Programme is presented in the appendix.

CHAPTER 2

THEORY AND METHOD OF SOLUTION

In this chapter the fourth order partial differential equation governing the motion of the pipe conveying fluid at velocity V is cast into simultaneous ordinary differential equations by finite element analysis using Galerkin's method. These equations are further rearranged into the standard eigenvalue problem and the complex eigenvalues obtained.

2.1 Finite Element Analysis :

Consider a horizontal pipe conveying fluid at a velocity V. The equation of motion for the small transverse vibration, can be found in Paidoussis and Issid [19]. Neglecting internal damping and gravity it becomes

$$(m_p + m_f)\frac{\partial^2 z}{\partial t^2} + 2m_f V \frac{\partial^2 z}{\partial x\,\partial t} + EI\frac{\partial^4 z}{\partial x^4}$$
$$+ (m_f V^2 - \bar{T} + \bar{P} A_f)\frac{\partial^2 z}{\partial x^2} = 0 \qquad (2.1)$$

and the boundary conditions are,

$$EI \frac{\partial^3 z}{\partial x^3} - T \frac{\partial z}{\partial x} + k_{d1} z = 0 \text{ or } z = 0 \text{ at } x = 0$$

$$EI \frac{\partial^2 z}{\partial x^2} - k_{t1} \frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = 0 \text{ at } x = 0$$

$$EI \frac{\partial^3 z}{\partial x^3} - T \frac{\partial z}{\partial x} - k_{d2} z = 0 \text{ or } z = 0 \text{ at } x = L$$

$$EI \frac{\partial^2 z}{\partial x^2} + k_{t2} \frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = 0 \text{ at } x = L \qquad \ldots \quad (2.2)$$

where

m_f = Mass of the fluid per unit length of the pipe.

m_p = Mass of the pipe per unit length.

V = Velocity of the fluid passing through the pipe.

A_f = Area of the passage through which the fluid passes.

T = Externally applied tension to the pipe.

P = Pressure of the fluid.

E = Modulus of elasticity of the pipe material.

I = Moment of inertia of the pipe.

L = Length of the pipe.

z = Transverse displacement of the pipe.

x = Axial co-ordinate of the pipe.

k_{d1}, k_{d2} = Displacement spring constants at left and right ends respectively.

k_{t1}, k_{t2} = Rotational spring constants at left and right ends respectively.

It is desired to solve this problem by finite-element analysis using Galerkin's method.

Let the beam have m nodes with m-1 elements as shown in Fig.1(b). The typical finite element is shown in Fig.1(c).

The transverse displacement z within the element is a function of space and time. It is assumed that over the element z varies as

$$z^{(e)}(x,t) = \sum_{i=1}^{f} N_i(x)\, z_i(t) = \lfloor N(x) \rfloor \{z(t)\}^{ne} \quad \ldots \quad (2.3)$$

where f is the number of degrees of freedom assigned to the element (e) and $z_i(t)$ are the discrete values at the nodes.

Applying Galerkin's criterion to Eqn (2.1) as done by Szabo and Lee [20] and Huebner [11], we get,

$$\int_0^1 N_i \left[(m_p + m_f) \frac{\partial^2 z^{(e)}}{\partial t^2} + 2m_f V \frac{\partial^2 z^{(e)}}{\partial x\, \partial t} + EI \frac{\partial^4 z^{(e)}}{\partial x^4} + (m_f V^2 - T + P A_f) \frac{\partial^2 z^{(e)}}{\partial x^2} \right] dx = 0$$

$$i = 1,2,\ldots\ldots,f \qquad (2.4)$$

The second, third and fourth terms of Eqn.(2.4) are integrated by parts to introduce natural boundary

condtions. At the same time this leads to less stringent requirements for the interpolation function $\lfloor N \rfloor$.

The second term gives,

$$\int_0^1 N_i \left(2m_f\, V \frac{\partial^2 z^{(e)}}{\partial x\, \partial t}\right) dx = N_i\, 2m_f\, V \frac{\partial z^{(e)}}{\partial t} \Big|_0^1$$

$$- \int_0^1 \frac{\partial N_i}{\partial x}\, 2m_f\, V \frac{\partial z^{(e)}}{\partial t}\, dx \quad \ldots \qquad (2.5)$$

The third term of equation (2.4) is integrated twice by parts, to get,

$$\int_0^1 N_i \left(EI \frac{\partial^4 z^{(e)}}{\partial x^4}\right) dx = N_i\, EI \frac{\partial^3 z^{(e)}}{\partial x^3} \Big|_0^1$$

$$- \frac{\partial N_i}{\partial x}\, EI \frac{\partial^2 z^{(e)}}{\partial x^2} \Big|_0^1 + \int_0^1 \frac{\partial^2 N_i}{\partial x^2}\, EI \frac{\partial^2 z^{(e)}}{\partial x^2}\, dx$$

$$\ldots \qquad (2.6)$$

The fourth term yields,

$$\int_0^1 N_i \left(m_f\, V^2 - T + P\, A_f\right) \frac{\partial^2 z^{(e)}}{\partial x^2} dx$$

$$= N_i \left(m_f\, V^2 - T + P\, A_f\right) \frac{\partial z^{(e)}}{\partial x} \Big|_0^1$$

$$- \int_0^1 \frac{\partial N_i}{\partial x} \left(m_f\, V^2 - T + P\, A_f\right) \frac{\partial z^{(e)}}{\partial x} dx \quad (2.7)$$

Substituting Eqns. (2.5), (2.6) and (2.7) into Eqn. (2.4), one gets,

$$\int_0^1 \{(m_p + m_f)\, N_i \frac{\partial^2 z^{(e)}}{\partial t^2} - 2m_f V \frac{\partial N_i}{\partial x} \frac{\partial z^{(e)}}{\partial t}$$

$$+ EI \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 z^{(e)}}{\partial x^2} - (m_f V^2 - T + P A_f) \frac{\partial N_i}{\partial x} \frac{\partial z^{(e)}}{\partial x} \}\, dx$$

$$+ N_i\, (2m_f V) \frac{\partial z^{(e)}}{\partial t}\Big|_0^1 + N_i\, EI \frac{\partial^3 z^{(e)}}{\partial x^3}\Big|_0^1$$

$$- \frac{\partial N_i}{\partial x} EI \frac{\partial^2 z^{(e)}}{\partial x^2}\Big|_0^1 + N_i\, (m_f V^2 - T + P A_f) \frac{\partial z^{(e)}}{\partial x}\Big|_0^1 = 0$$

$$i = 1, 2, \ldots, f \qquad (2.8)$$

Substituting the value of $z^{(e)}$ from Eqn. (2.3) in Eqn.(2.8), one gets,

$$\int_0^1 [\, (m_p + m_f)\{ N \} \lfloor N \rfloor \{ \frac{\partial^2 z}{\partial t^2} \}^{ne}$$

$$- 2m_f V \{ N' \} \lfloor N \rfloor \{ \frac{\partial z}{\partial t} \}^{ne}$$

$$+ EI \{ N'' \} \lfloor N'' \rfloor \{ z \}^{ne}$$

$$- (m_f V^2 - T + P A_f) \{ N' \} \lfloor N' \rfloor \{ z \}^{ne} \,]\, dx$$

$$= - \{N\}\, EI \frac{\partial^3 z^{(e)}}{\partial x^3}\Big|_0^1$$

$$- \{N\}\, (m_f V^2 - T + P A_f) \frac{\partial z^{(e)}}{\partial x}\Big|_0^1$$

$$- \{ N \}\, (2m_f V) \frac{\partial z^{(e)}}{\partial t}\Big|_0^1 + \{ N' \}\, EI \frac{\partial^2 z}{\partial x^2}\Big|_0^1$$

$$\ldots \qquad (2.9)$$

Eqn. (2.9) can be written in the following form,

$$[m]^{(e)}\{\ddot{z}\}^{ne} + [c]^{(e)}\{\dot{z}\}^{ne} + [k]^{(e)}\{z\}^{ne} =$$

$$-\left\{\begin{array}{l} N_1 \; EI \dfrac{\partial^3 z^{(e)}}{\partial x^3} \Big|_0^1 \\ N_2 \; EI \dfrac{\partial^3 z^{(e)}}{\partial x^3} \Big|_0^1 \\ \vdots \\ N_f \; EI \dfrac{\partial^3 z^{(e)}}{\partial x^3} \Big|_0^1 \end{array}\right\}$$

$$-\left\{\begin{array}{l} N_1 \; (m_f V^2 - T + P A_f) \dfrac{\partial z^{(e)}}{\partial x} \Big|_0^1 \\ N_2 \; (m_f V^2 - T + P A_f) \dfrac{\partial z^{(e)}}{\partial x} \Big|_0^1 \\ \vdots \\ N_f \; (m_f V^2 - T + P A_f) \dfrac{\partial z^{(e)}}{\partial x} \Big|_0^1 \end{array}\right\}$$

$$-\left\{\begin{array}{l} N_1 \; (2m_f V) \dfrac{\partial z^{(e)}}{\partial t} \Big|_0^1 \\ N_2 \; (2m_f V) \dfrac{\partial z^{(e)}}{\partial t} \Big|_0^1 \\ \vdots \\ N_f \; (2m_f V) \dfrac{\partial z^{(e)}}{\partial t} \Big|_0^1 \end{array}\right\} + \left\{\begin{array}{l} N_1' \; EI \dfrac{\partial^2 z^{(e)}}{\partial x^2} \Big|_0^1 \\ N_2' \; EI \dfrac{\partial^2 z^{(e)}}{\partial x^2} \Big|_0^1 \\ \vdots \\ N_f' \; EI \dfrac{\partial^2 z^{(e)}}{\partial x^2} \Big|_0^1 \end{array}\right\} \quad \ldots . \quad (2.10)$$

where

$$[m]^{(e)} = \text{Mass matrix of the element}$$

$$= \int_{x=0}^{l} (m_p + m_f) \{N\} \lfloor N \rfloor \, dx \qquad (2.11)$$

$$[c]^{(e)} = \text{Damping matrix of the element}$$

$$= \int_{x=0}^{l} - 2m_f V \{N'\} \lfloor N \rfloor \, dx \qquad (2.12)$$

$$[k]^{(e)} = \text{Stiffness matrix of the element}$$

$$= \int_{x=0}^{l} [EI \{N''\} \lfloor N'' \rfloor - (m_f V^2 - T + P A_f)\{N'\} \lfloor N' \rfloor] \, dx \quad \ldots\ldots \quad (2.13)$$

According to Szabo and Lee [20] the column matrices, on the right hand side of the Eqns. (2.10), are the measures of the error of discretization of the finite-element method. But it is these matrices that introduce the natural boundary conditions of the problem and as such must be taken into consideration. This fact was recognized by Huebner [11] , but there is still some confusion when elasticity problems are considered.

While choosing the shape function $\lfloor N \rfloor$ the following conditions must be met to ensure monotonic convergence, Huebner [11] . All uniform states of z and its partial derivatives upto the highest order appearing in Eqn.(2.8)

should have a representation in $z^{(e)}$ when in the limit the element size shrinks to zero. Also at the elemental interfaces z and any of its partial derivative one order less than the highest appearing in integrants of Eqn. (2.8) must be continuous. Elements satisfying the first requirement are known as complete and elements satisfying the second requirement are known as compatible in the literature.

The following shape function commonly used for beam elements has been taken for this problem as it satisfies the above requirements, Desai and Able [6]

$$\lfloor N \rfloor = \lfloor \xi_1^2 (3 - 2\xi_1) \quad \xi_1^2 \ \xi_2 l \quad \xi_2^2 (3 - 2\xi_2) \quad -\xi_1 \ \xi_2^2 l \rfloor \qquad \ldots. \quad (2.14)$$

where $\xi_1 = 1 - \frac{x}{l}$, $\xi_2 = \frac{x}{l}$

and l = length of the beam element

Substituting the shape function (2.14) into equations (2.11), (2.12) and (2.13) and integrating, one gets,

$$[m]^{(e)} = \frac{(m_p + m_f)\ l}{420} \quad X$$

$$\begin{bmatrix} 156 & 22\,l & 54 & -13\,l \\ & 4\,l^2 & 13\,l & -3\,l^2 \\ & & 156 & -22\,l \\ \text{Symmetric} & & & 4\,l^2 \end{bmatrix} \qquad (2.15)$$

$$[c]^{(e)} = \frac{2m_f V}{60}\begin{bmatrix} 30 & 6l & +30 & -6l \\ -6l & 0 & 6l & -l^2 \\ -30 & -6l & -30 & 6l \\ 6l & l^2 & -6l & 0 \end{bmatrix}$$

$$\ldots \quad (2.16)$$

and

$$[k]^{(e)} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{Symmetric} & & & 4l^2 \end{bmatrix} + \left(\frac{T - P A_f - m_f V^2}{30l}\right)\begin{bmatrix} 36 & 3l & -36 & 3l \\ & 4l^2 & -3l & -l^2 \\ & & 36 & -3l \\ \text{Symmetric} & & & 4l^2 \end{bmatrix}$$

$$\ldots \quad (2.17)$$

and Eqns. (2.10) become

$$[m]^{(e)}\{\ddot{z}\}^{ne} + [c]^{(e)}\{\dot{z}\}^{ne} + [k]^{(e)}\{z\}^{ne} = -\begin{Bmatrix} -EI\left.\frac{\partial^3 z^{(e)}}{\partial x^3}\right|_j \\ 0 \\ +EI\left.\frac{\partial^3 z^{(e)}}{\partial x^3}\right|_k \\ 0 \end{Bmatrix}$$

$$-\begin{Bmatrix} -(m_f V^2 - T + P A_f)\,\theta_j \\ 0 \\ +(m_f V^2 - T + P A_f)\,\theta_k \\ 0 \end{Bmatrix} - \begin{Bmatrix} -2m_f V \dot{z}_j \\ 0 \\ 2m_f V \dot{z}_k \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -EI\left.\frac{\partial^2 z^{(e)}}{\partial x^2}\right|_j \\ 0 \\ EI\left.\frac{\partial^2 z^{(e)}}{\partial x^2}\right|_k \end{Bmatrix}$$

$$\ldots \quad (2.18)$$

For the whole domain of the problem, Eqns. (2.18), become

$$[M]_{BB}\{\ddot{z}\}^n + [C]_{BB}\{\dot{z}\}^n + [K]_{BB}\{z\}^n =$$

$$\left\{\begin{matrix} EI \left.\dfrac{\partial^3 z}{\partial x^3}\right|_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -EI \left.\dfrac{\partial^3 z}{\partial x^3}\right|_m \\ 0 \end{matrix}\right\} + \left\{\begin{matrix} (m_f V^2 - T + P A_f)\,\theta_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -(m_f V^2 - T + P A_f)\,\theta_m \\ 0 \end{matrix}\right\}$$

$$+\left\{\begin{matrix} 2m_f V \dot{z}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2m_f V \dot{z}_m \\ 0 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ -EI \left.\dfrac{\partial^2 z}{\partial x^2}\right|_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ EI \left.\dfrac{\partial^2 z}{\partial x^2}\right|_m \end{matrix}\right\}. \qquad (2.19)$$

where $[M]_{BB}$, $[C]_{BB}$, $[K]_{BB}$ are the assembled mass, damping and stiffness matrices, respectively, before the boundary conditions are applied. For computational ease, Eqns. (2.19) are rearranged as follows:

$$[M]_{BB}\{\ddot{z}\}^{n}+[C]_{BB}\{\dot{z}\}-\begin{Bmatrix} 2m_f V \dot{z}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -2m_f V \dot{z}_m \\ 0 \end{Bmatrix}$$

$$+[K]_{BB}\{z\}^{n}-\begin{Bmatrix} (m_f V^2 - T + P A_f)\,\theta_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -(m_f V^2 - T + P A_f)\,\theta_m \\ 0 \end{Bmatrix}=$$

$$\begin{Bmatrix} EI \frac{\partial^3 z}{\partial x^3}\Big|_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -EI \frac{\partial^3 z}{\partial x^3}\Big|_m \\ 0 \end{Bmatrix}+\begin{Bmatrix} 0 \\ -EI \frac{\partial^2 z}{\partial x^2}\Big|_1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ +EI \frac{\partial^2 z}{\partial x^2}\Big|_m \end{Bmatrix} \qquad (2.20)$$

It should be noted that $2m_f V \dot{z}_1$ and $2m_f V \dot{z}_m$ will go to $C_{1,1}$ and $C_{2m-1,\,2m-1}$ respectively, whereas

$(m_f V^2 - T + P A_f)\,\theta_1$ and $(m_f V^2 - T + P A_f)\,\theta_m$ will go to $K_{1,2}$ and $K_{2m-1,\,2m}$ respectively.

After applying boundary condition Eqn. (2.20) can be written in the form,

$$[M]\,\{\ddot{z}\}^n + [C]\,\{\dot{z}\}^n + [K]\,\{z\}^n = \{0\} \qquad (2.21)$$

2.2 Method of Solution :

The solution of these homogeneous set of differential equations is obtained as detailed in Meirovitch [14] . It should be noted that it is not necessary, as Meirovitch [14] requires that [M] , [C] , [K] matrices be symmetric, see Frazer, Duncan and Collar [7] . In the following, Eqns. (2.21) are taken as n equations.

Using generalised velocities $\{\dot{z}\}$ as auxiliary variables, n second order ordinary differential equations (2.21) are converted to a set of 2n first order ordinary differential equations, Meirovitch [14]

$$[\bar{M}]\,\{\dot{y}(t)\} + [\bar{K}]\,\{y(t)\} = \{Y(t)\} \qquad (2.22)$$

where

$$\{y(t)\} = \begin{Bmatrix} \{\dot{z}(t)\} \\ \{z(t)\} \end{Bmatrix} \text{and } \{Y(t)\} = \{0\} \qquad (2.23)$$

are column matrices consisting of 2n elements representing

generalized coordinates and generalized forces respectively, and

$$[\bar{M}] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \text{ and } [\bar{K}] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \qquad (2.24)$$

are real matrices of order 2n.

Now we have the differential equation which in the matrix form is,

$$[\bar{M}]\ \{\dot{y}(t)\} + [\bar{K}]\ \{y(t)\} = \{0\} \qquad (2.25)$$

Let $\{y(t)\} = e^{\alpha t}\ \{\phi\}$ (2.26)

where ϕ represents a vector consisting of 2n constant elements. Substitution of Eqn. (2.26) in Eqns. (2.25) leads to the eigenvalue problem.

$$\alpha\ [\bar{M}]\{\phi\} + [\bar{K}]\ \{\phi\} = \{0\} \qquad (2.27)$$

which can be written as

$$[D]\ \{\phi\} = \frac{1}{\alpha}\ \{\phi\} \qquad (2.28)$$

where

$$[D] = -[\bar{K}]^{-1}\ [\bar{M}] = \begin{bmatrix} [0] & [I] \\ -[K]^{-1}\ [M] & -[K]^{-1}\ [C] \end{bmatrix} \qquad \ldots \quad (2.29)$$

This plays the role of the dynamical matrix.

The eigenvalues of the above dynamical matrix (2.29) will in general be complex. Their real parts determine the stability of the system and the imaginary parts give the frequency of vibration. If the real part is negative, the system will be stable. The system will have oscilatory instability (flutter) if real part is positive and imaginary part is non-zero. On the other hand if this imaginary part is zero, the system will have buckling instability.

In this formulation the effects of various types of geometrical and natural boundary conditions as well as the effect of axial load, fluid pressure and velocity on the pipe stability can be studied very easily.

A computer programme for calculating the complex eigenvalue and eigenvectors of a dynamical matrix was written, Meirovitch [14] . Later on an in-built library subroutine was used to calculate the complex eigenvalues.

CHAPTER 3

RESULTS AND DISCUSSION

This chapter deals with the numerical results obtained for pipes conveying fluid with different boundary conditions. Complex frequencies of the lowest four modes have been found as a function of the velocity of the fluid flow through the pipe. From the analytical point of view the mechanism of instability has been discussed. The results are given in terms of the dimensionless parameters usually used in the literature.

3.1 Dimensionless Parameters :

$$\beta = \text{Dimensionless mass} = \frac{m_f}{m_f + m_p}$$

$$u = \text{Dimensionless fluid flow velocity} = \frac{VL}{\sqrt{EI/m_f}}$$

$$Re(\omega) = \text{Real part of dimensionless frequency} = \frac{\omega_R}{\sqrt{EI/((m_f + m_p)\ L^4)}}$$

$$Im(\omega) = \text{Imaginary part of dimensionless frequency} = \frac{\omega_I}{\sqrt{EI/((m_f + m_p)\ L^4)}}$$

α_d = Dimensionless displacement spring factor $= \frac{k_d L^3}{EI}$

α_t = Dimensionless rotational spring factor $= \frac{k_t L}{EI}$

3.2 Number of Finite Elements :

To begin with one must decide the number of finite elements to be used in the analysis. First, for every boundary condition results were obtained for a few velocities with the number of elements varying between 3 and 7. It was found that after five elements there was hardly any change in the results in all the cases. The sample results for pinned-pinned and cantilever pipe are shown in Table 1. All results reported in this dissertations are for the pipe with five finite elements.

3.3 Complex Frequencies of Pipes:

The calculated frequencies of the pipe problem for different boundary conditions are plotted in Argand diagrams. The imaginary part of the eigenvalue Im(ω) which gives the frequency of the vibration is plotted on the abcissa and the real part Re(ω) is plotted on the ordinate. The various parameters are u, β, α_d and α_t.

3.3.1 Pinned-Pinned Pipe

Figure (2) shows the dimensionless complex frequencies of a pinned-pinned pipe with dimensionless flow

TABLE 1

Number of Elements Vs Frequencies

(a) Pinned - Pinned Pipe ($\beta = 0.5$)

No. of Elements	u = 3.0				u = 6.0			
	First Mode		Second Mode		First Mode		Second Mode	
	Re(ω)	Im(ω)	Re(ω)	Im(ω)	Re(ω)	Im(ω)	Re(ω)	Im(ω)
3	0	2.814	0	35.925	9.699	0	0	18.924
4	0	2.793	0	35.466	9.383	0	0	17.757
5	0	2.787	0	35.329	9.285	0	0	17.404
6	0	2.785	0	35.279	9.247	0	0	17.269
7	0	2.784	0	35.256	9.231	0	0	17.210

Number of Elements Vs Frequencies

(b) Cantilever Pipe ($\beta = 0.5$)

No. of Elements	u = 2.0				u = 6.0			
	First Mode		Second Mode		First Mode		Second Mode	
	Re(ω)	Im(ω)	Re(ω)	Im(ω)	Re(ω)	Im(ω)	Re(ω)	Im(ω)
3	- 2.861	2.692	- 2.826	21.047	- 10.963	2.867	- 6.095	12.207
4	- 2.860	2.691	- 2.801	20.988	- 10.937	2.972	- 5.974	11.974
5	- 2.860	2.691	- 2.793	20.970	- 10.930	3.003	- 5.933	11.898
6	- 2.859	2.691	- 2.790	20.963	- 10.927	3.015	- 5.917	11.868
7	- 2.859	2.691	- 2.789	20.960	- 10.926	3.020	- 5.910	11.855

velocity u as a parameter for $\beta = 0.5$. It is noted that the results match with those obtained by the classical method, Paidoussis and Issid [19]. Complete agreement was observed for the case $\beta = 0.2$ also, though it is not presented here.

3.3.2 Fixed-Fixed Pipe

Figure (3) shows the dimensionless complex frequencies of a fixed-fixed pipe with dimensionless flow velocity u as a parameter for $\beta = 0.5$. These results also match with those by classical methods, Paidoussis and Issid [19].

3.3.3 Cantilever Pipe

Figure (4) shows the dimensionless complex frequencies for a cantilever pipe with dimensionless flow velocity u as a parameter for $\beta = 0.2$. Here again our results agree with those obtained by the classical method, Gregory and Paidoussis [8]. The results for the cantilever pipe for $\beta = 0.295$ also match with the classical one in the same reference.

Here it is very important to note that the column matrices $\{ N_i \, 2 m_f \, V \frac{\partial z}{\partial t}^{(e)} \Big|_0^1 \}$ and $\{ N_i \, (m_f V^2 - \bar{T} + P A_f) \frac{\partial z}{\partial x}^{(e)} \Big|_0^1 \}$ found in Eqn. (2.8) must be taken into account in damping

and stiffness matrices, respectively. In the analysis these can not be neglected as suggested in the work of Szabo and Lee [20]. Results obtained neglecting these terms are shown in Fig (5). This figure clearly indicates the large difference from the true results, (compare Fig. (4)).

3.3.4 Pipe with One End Fixed and the Other Supported by a Displacement Spring

The boundary conditions for this case are

$$z = 0 \quad \text{and} \quad \frac{\partial z}{\partial x} = 0 \qquad ; \text{ at } x = 0$$

$$EI \frac{\partial^3 z}{\partial x^3} - k_{d2} z = 0 \quad \text{and} \quad EI \frac{\partial^2 z}{\partial x^2} = 0 \qquad ; \text{ at } x = L \tag{3.1}$$

Figure (5) shows the dimensionless complex frequencies for the above mentioned system with dimensionless flow velocity u as a parameter for $\beta = 0.6$ and $\alpha_t = 100$. From the figure it is clear that the results are the same as those of the classical method, Chen [4]. Results were also found to match for the case $\beta = 0.2$ and $\alpha_d = 10$.

Here also a large difference in the results was observed when the column matrices $\{ N_i \, 2 m_f V \frac{\partial z}{\partial t}^{(e)} \Big|_0^1 \}$ and $\{ N_i (m_f V^2 - T + P A_f) \frac{\partial z}{\partial x}^{(e)} \Big|_0^1 \}$ were not added in damping and stiffness matrices.

3.3.5 Pipe with one End Fixed and the other Supported by a Displacement and a Rotational Spring

Boundary conditions for this case are

$$z = 0 \quad \text{and} \quad \frac{\partial z}{\partial x} = 0 \qquad ; \quad \text{at } x = 0$$

$$EI \frac{\partial^3 z}{\partial x^3} - k_{d2}\, z = 0 \quad \text{and} \quad EI \frac{\partial^2 z}{\partial x^2} + k_{t2} \frac{\partial z}{\partial x} = 0 \;;\; \text{at } x = L \qquad \ldots \quad (3.2)$$

Figure (7) shows the dimensionless complex frequencies for the system mentioned above with dimensionless flow velocity u as a parameter for $\beta = 0.6$, $\alpha_d = 50$, $\alpha_t = 25$, and it is found that the results obtained by finite element method match with those by the classical method, Lin and Chen [12]. Agreement was also observed for the case of $\beta = 0.2$, $\alpha_d = 10$ and $\alpha_t = 5$.

It is to be noted again that column matrices $\{ N_i \; 2m_f \, V \frac{\partial z}{\partial t}^{(e)} \Big|_0^1 \}$ and $\{ N_i \; (m_f V^2 - T + P A_f) \frac{\partial z}{\partial x}^{(e)} \Big|_\theta^1 \}$ have to be added to damping and stiffness matrices, otherwise large discrepencies are found in the results.

3.3.6 Pipe with One End Fixed and the Other Pinned

Boundary conditions for this case are

$$z = 0 \quad \text{and} \quad \frac{\partial z}{\partial x} = 0 \qquad ; \text{ at } x = 0$$

$$z = 0 \quad \text{and} \quad EI \frac{\partial^2 z}{\partial x^2} = 0 \qquad ; \text{ at } x = L \qquad (3.3)$$

Figure (8) shows the dimensionless complex frequencies with dimensionless flow velocity u as a parameter for $\beta = 0.5$. In the figure it is to be noted that, with increasing flow velocity, the frequency of the system decreases and the frequency of the first mode becomes zero at $u \approx 4.5$ which is the first critical flow velocity for buckling. But at $u \approx 7.7$ the system regains stability in the first mode. At slightly higher flow velocity at $u \approx 7.9$ the first and the second mode loci coalesce on the $[\mathrm{Im}(\omega)]$ - axis, and coupled - mode flutter with first and second modes occurs. As the flow velocity u is further increased the frequency of vibration vanishes at $u \approx 11.1$ for the first mode. This is the second critical velocity (buckling). At a slightly higher velocity $u \approx 11.2$ the coupled mode flutter occurs in the second and third modes.

In the case of this boundary condition it was noted that the critical flow velocity does not depend upon β as in the case of pinned-pinned pipes and fixed-fixed pipes, Paidoussis and Issid [19]. Here the critical velocity is 4.5 and is associated with buckling. It is interesting to note that this value lies between those of pinned-pinned (π) and fixed-fixed pipes (2π).

3.3.7 Pipe with Both the Ends Supported by a Displacement and a Rotational Spring (Symmetric Boundaries)

Boundary conditions for this case are

$$EI \frac{\partial^3 z}{\partial x^3} + k_{d1} z = 0 \quad \text{and} \quad EI \frac{\partial^2 z}{\partial x^2} - k_{t1} \frac{\partial z}{\partial x} = 0 \quad ; \text{ at } x = 0$$

$$EI \frac{\partial^3 z}{\partial x^3} - k_{d2} z = 0 \quad \text{and} \quad EI \frac{\partial^2 z}{\partial x^2} + k_{t2} \frac{\partial z}{\partial x} = 0 \quad ; \text{ at } x = L$$

$$(3.4)$$

Figure (9) shows, for $\beta = 0.2$, $\alpha_d = 10$ and $\alpha_t = 5$, the dimensionless complex frequencies for the above system with dimensionless flow velocity u as a parameter. With the increase in flow velocity u the frequency decreases and becomes zero at $u \approx 4.25$ in the first mode, which is the first critical flow velocity for buckling. At flow velocity $u \approx 4.8$ the frequency becomes zero in the second mode. As the flow velocity is increased further, the loci of the first and second modes coalesce on the $[\,\mathrm{Re}(\omega)\,]$ - axis and leave the axis at symmetrical points, representing the coupled-mode flutter with the first and second modes.

3.3.8 Pipe with One End Supported by a Displacement Spring and a Rotational Spring and the Other End by a Displacement Spring Only (Non-Symmetric Boundaries)

Boundary conditions for this case are

$$EI\frac{\partial^3 z}{\partial x^3} + k_{d1}\, z = 0 \quad \text{and} \quad EI\frac{\partial^2 z}{\partial x^2} - k_{t1}\frac{\partial z}{\partial x} = 0 \quad ; \quad \text{at} \quad x = 0$$

$$EI\frac{\partial^3 z}{\partial x^3} - k_{d2}\, z = 0 \quad \text{and} \quad EI\frac{\partial^2 z}{\partial x^2} = 0 \quad ; \quad \text{at} \quad x = L \qquad \ldots \quad (3.5)$$

Figure (10) shows, for $\beta = 0.2$, $\alpha_d = 10$ and $\alpha_t = 5$, the dimensionless complex frequencies with dimensionless flow velocity u as a parameter. As usual in the beginning the frequency of vibration decreases with increase in fluid flow velocity. At $u \approx 4$, the pipe becomes unstable due to flutter in the second mode and is the first critical velocity for this case. With further increase in the flow velocity at $u \approx 4.25$ the frequency become zero in the first mode but still the system remains unstable in the second mode with flutter. The first and third modes always remain stable but the second mode is still unstable (flutter), moreover at flow velocity $u = 10.2$ there occurs flutter instability in the fourth mode as well.

3.3.9 Pipe with One End Supported by a Displacement and a Rotational Spring and the Other End by a Rotational Spring only (Non-Symmetric Boundaries)

Boundary conditions for this case are

$$EI\frac{\partial^3 z}{\partial x^3} + k_{d1}\, z = 0 \quad \text{and} \quad EI\frac{\partial^2 z}{\partial x^2} - k_{t1}\frac{\partial z}{\partial x} = 0 \quad ; \quad \text{at} \quad x = 0$$

$$EI\frac{\partial^3 z}{\partial x^3} = 0 \quad \text{and} \quad EI\frac{\partial^2 z}{\partial x^2} + k_{t2}\frac{\partial z}{\partial x} = 0 \quad ; \quad \text{at} \quad x = L \qquad \ldots \quad (3.6)$$

Figure (11) shows for $\beta = 0.2$, $\alpha_d = 10$ and $\alpha_t = 5$ the dimensionless complex frequencies with dimensionless flow velocity u as a parameter. It is seen that this system is always unstable (flutter) in the second mode.

3.4 The Mechanism of Instability:

The mechanism of buckling and flutter instabilities of a pipe conveying fluid can be explained as follows:

3.4.1 Buckling Instabilities

It can be seen that the term $(m_f V^2 - T + P A_f)\frac{\partial^2 z}{\partial x^2}$ in the governing equation of motion act as an equivalent compressive force. As the fluid flow velocity increases, this compressive force increase thus reducing natural frequencies of the system. It may become zero and may lead to the buckling instability. Critical flow velocity may be found by the application of Euler's method of equilibrium, Paidoussis and Issid [19] . But these results based on a static analysis may not hold true for the dynamic case, Paidoussis and Issid [19].

Physically one can also visualize $m_f V^2 \frac{\partial^2 z}{\partial x^2}$ as the centrifugal force. When this force increases due to flow velocity and dominates over the flexural restoring force, the pipe buckles.

3.4.2 Flutter Instability

This can be explained by energy considerations. When the energy transfer from fluid to the pipe and vice versa exactly balances in one complete cycle of oscillation, the condition is one of neutral instability, that is dynamic equilibrium. But when the former is more than the latter, then the amplitude of oscillation increases without limit. But when the energy transfer by the pipe to the fluid is more than the energy transfer from the fluid to the pipe, the oscillations are damped.

Analytically flutter may be attributed to the Coriolis component in the governing equation of the system. If the frequencies of two modes become equal, the flutter is called a coupled - mode flutter and oscillations are of the form at exp $(i\omega t)$, Paidoussis and Issid[19] . So far these forms of oscillations were observed in conservative systems alone. Now it is observed that they can exist in non-conservative systems as well. For example see Fig. (9) for pipes with symmetrical spring supports.

CHAPTER 4

CONCLUSIONS

Based on the analysis of chapter 3 , I have reached the following conclusions.

(1) The results of pipe problems by the finite element method are found to match with the results obtained by the classical method for the cases of pinned-pinned, fixed-fixed, cantilever, cantilever with free end supported by displacement spring and cantilever with the free end supported by displacement and rotational springs. This shows that finite element equations of motion developed are correct.

(2) The column matrices $\{ N_i \, 2m_f \, V \frac{\partial z}{\partial t}^{(e)} \Big|_0^1 \}$ and $\{N_i \, (m_f \, V^2 - T + P \, A_f) \frac{\partial z}{\partial x}^{(e)} \Big|_0^1 \}$, found after applying Galerkin's Method to the governing differential equation of motion, must be included in the damping and stiffness matrices respectively. These column matrices can not be neglected as is suggested by the work of Szabo and Lee [20].

(3) The stability of the pipe with its upstream end fixed and the downstream end pinned has been studied. It is found that the system looses stability by buckling at a velocity $u \approx 4.5$ in the first mode and this critical velocity is independent of mass ratio β, as in the case of pinned-pinned pipes. Flutter is also possible at higher velocities in this case also.

From the stability point of view this system should be designed such that the velocity of the fluid passing through the pipe does not exceed $u = 4.5$.

(4) It is shown in Figures (9, 10, 11) that the pipe may loose its stability by buckling, flutter or both (coupled-mode), depending upon the magnitude of the displacement as well as the rotational spring constants. One thing is interesting to note that for the case when the spring supports are symmetric the instability occures in buckling first, whereas in the case of non-symmetric spring supports the likelyhood of instability is by flutter in the second mode.

(5) The finite element method for solving pipe problems has been found to be very accurate. By this method problems with different boundary conditions can be solved very easily. The results of the desired problem can be obtained just by changing the input data of the computer programme.

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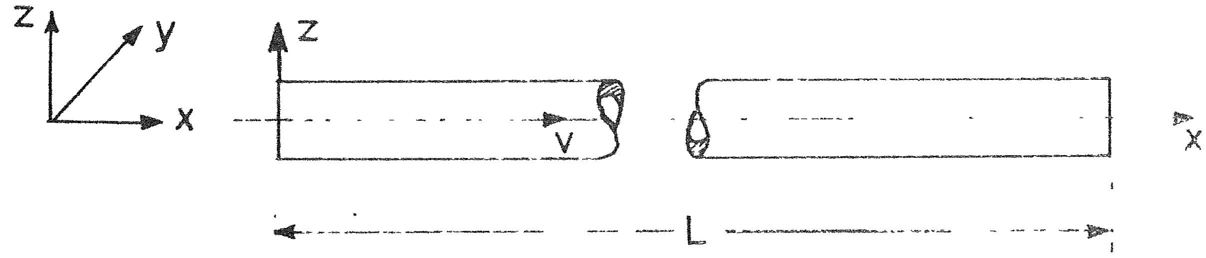
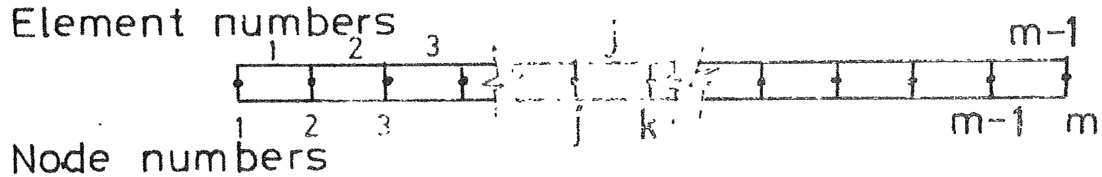
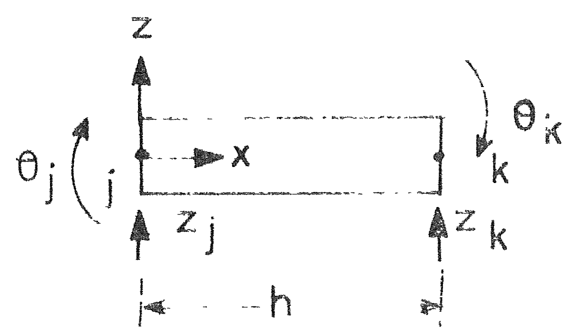


(a) Pipe conveying fluid at velocity V.



(b) Nodes and Finite Elements of the pipe.



(c) Typical Finite Element.

Fig. 1 Pipe and its finite element representation

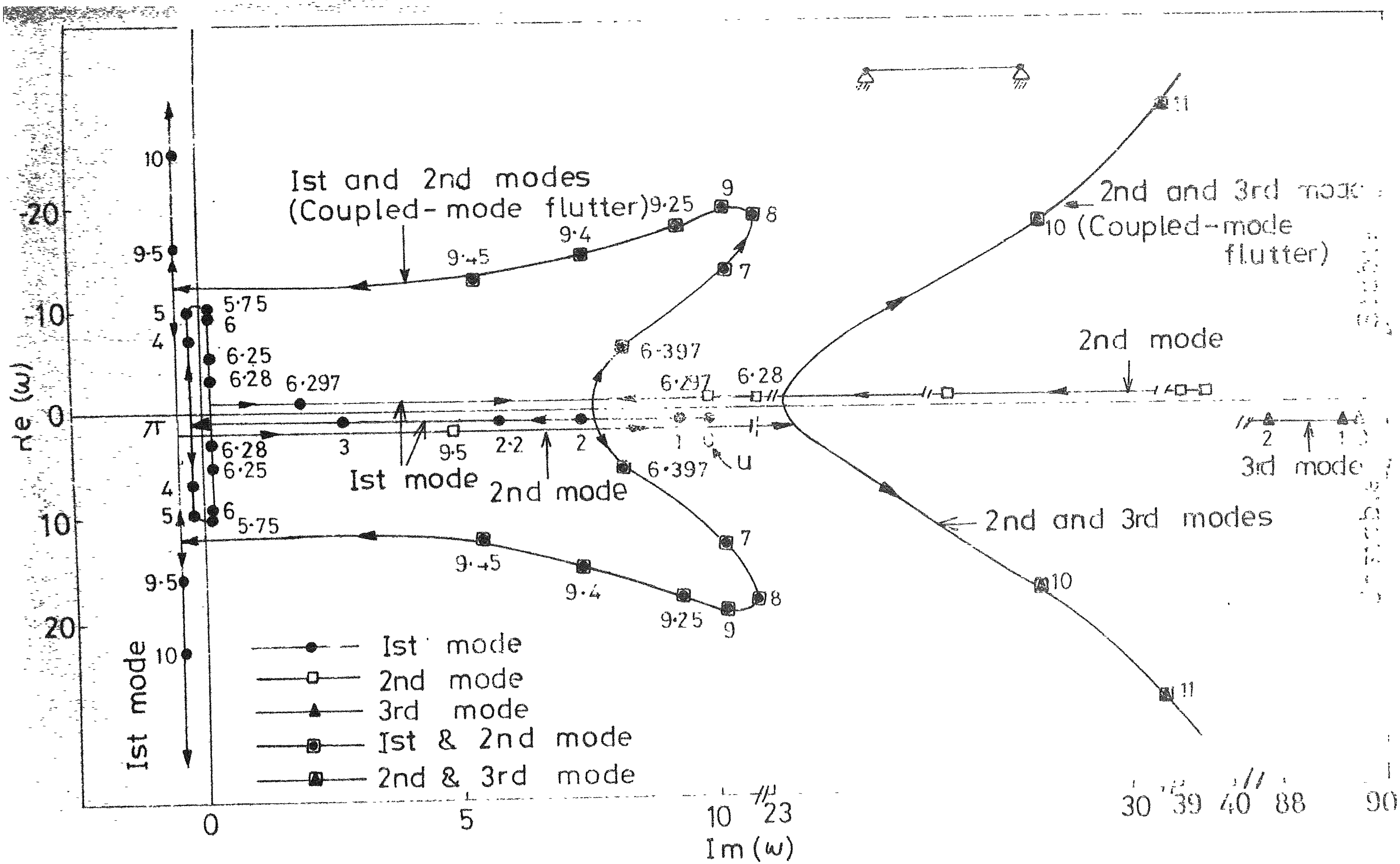


Fig.2 Dimensionless complex frequency diagram of pinned-pinned pipe, as function of dimensionless velocity u, β = 0·5, NEL = 5

T = 0, P = 0

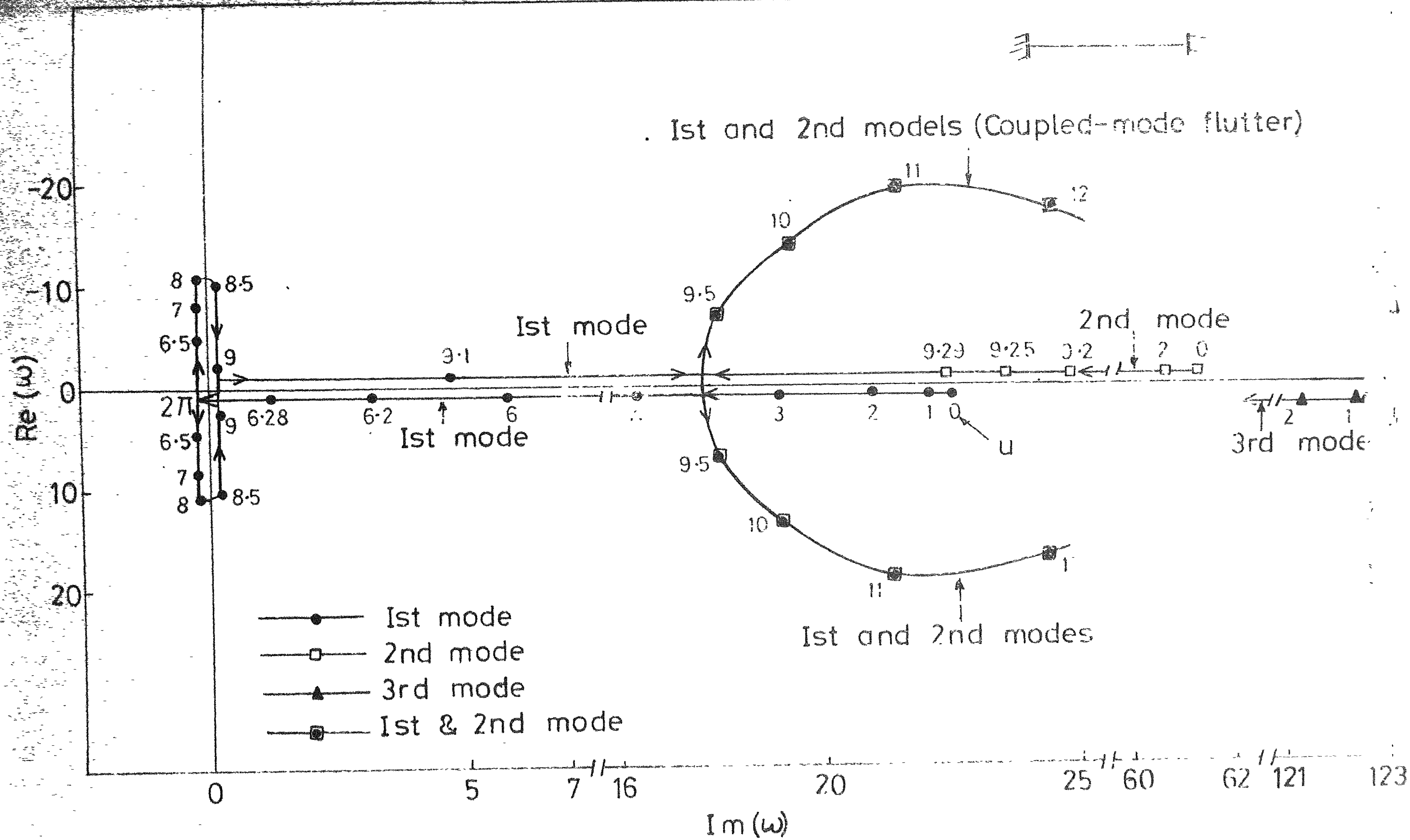


Fig. 3 Dimensionless complex frequency diagram of clamped-clamped pipe, as function of dimensionless velocity u, $\beta=0{\cdot}5$, NEL=5

$T = 0, \; P = 0$

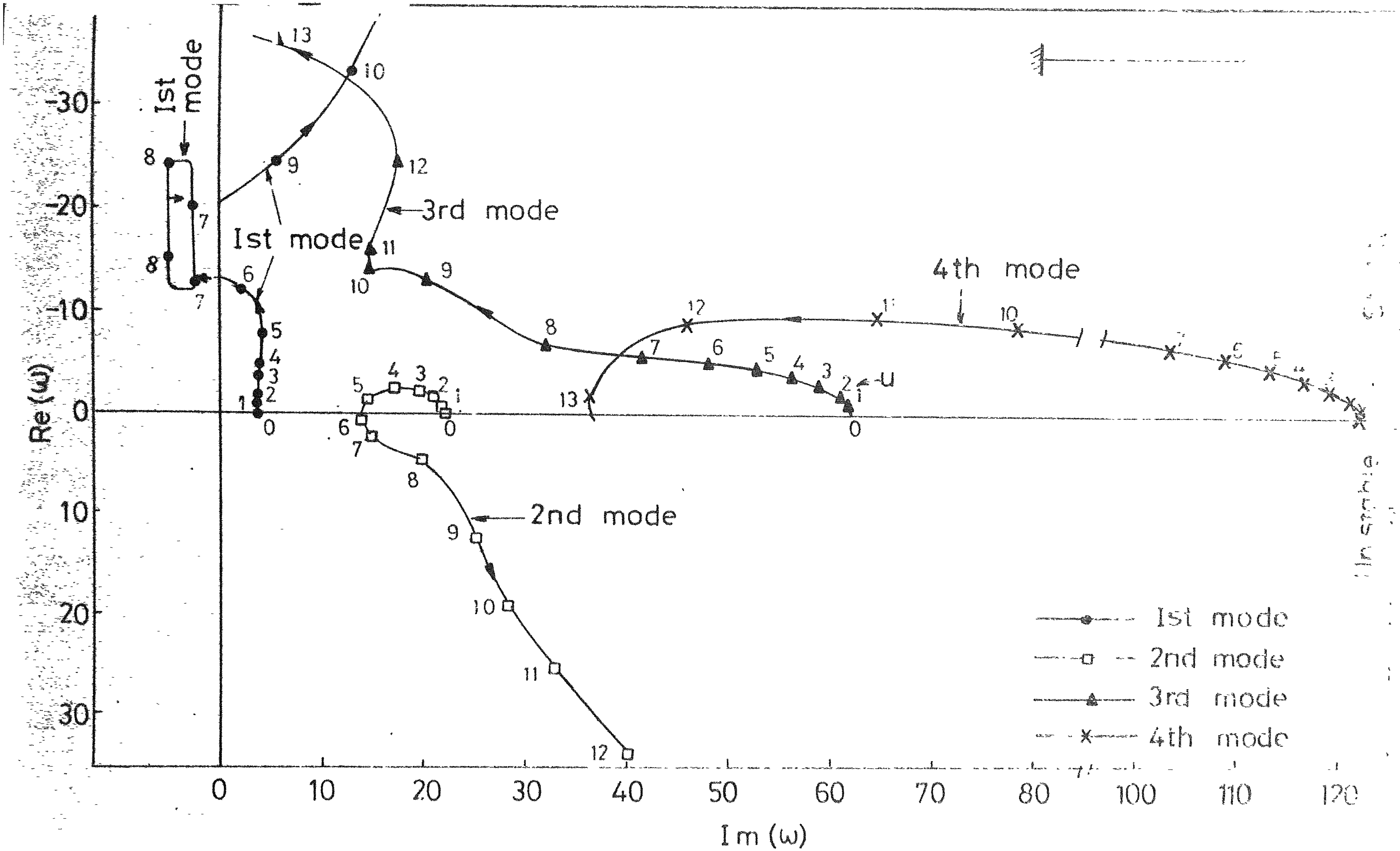


Fig.4 Dimensionless complex frequency of cantilever pipe, as a function of dimensionless velocity u, $\beta = 0.2$, NEL=5, $\tau = 0$, $\rho = 0$

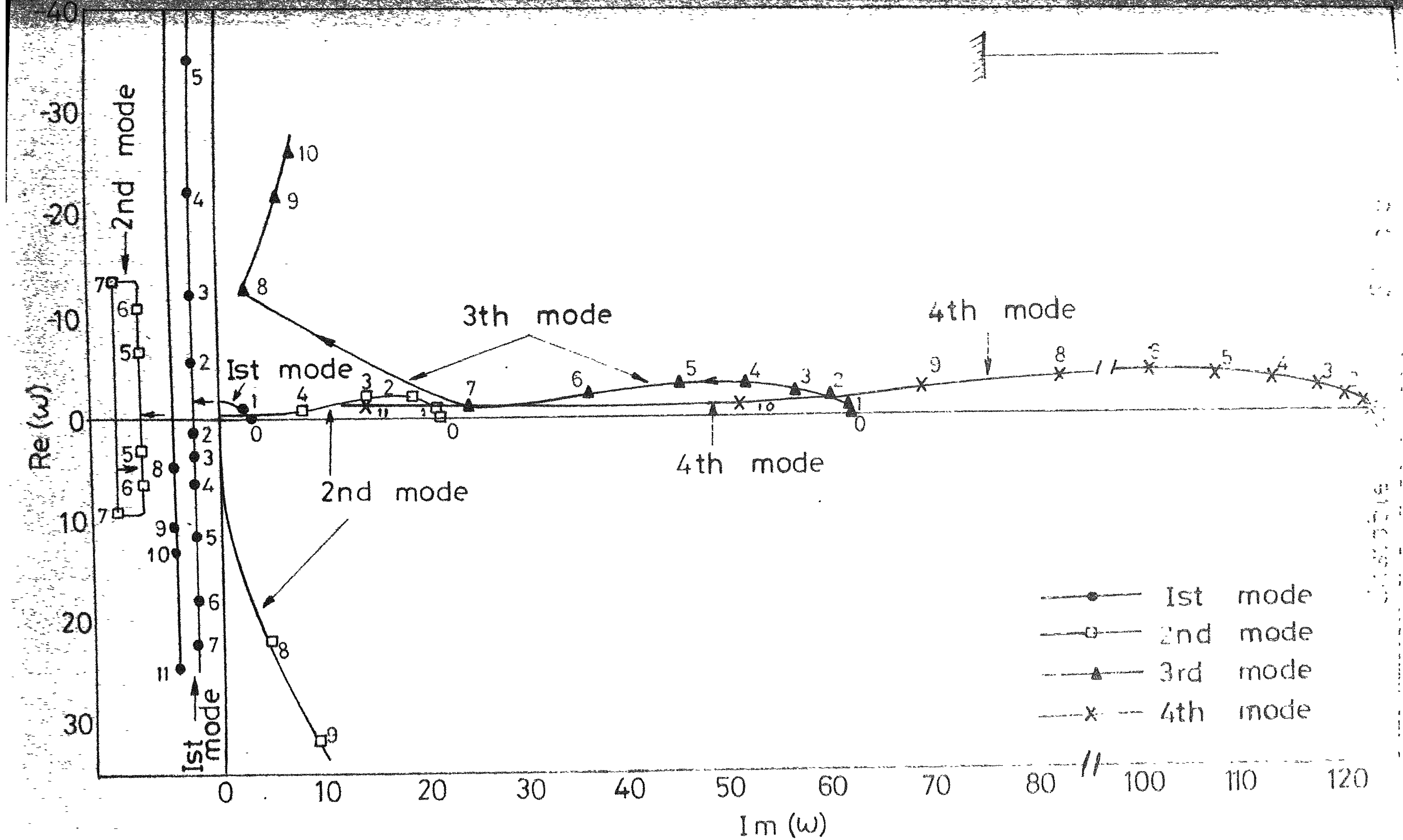


Fig.5 Dimensionless complex frequency of cantilever pipe, as a function of dimensionless velocity u, β=0.2, NEL=5 (Case without modifying stiffness and damping matrices)

T=0, P=0

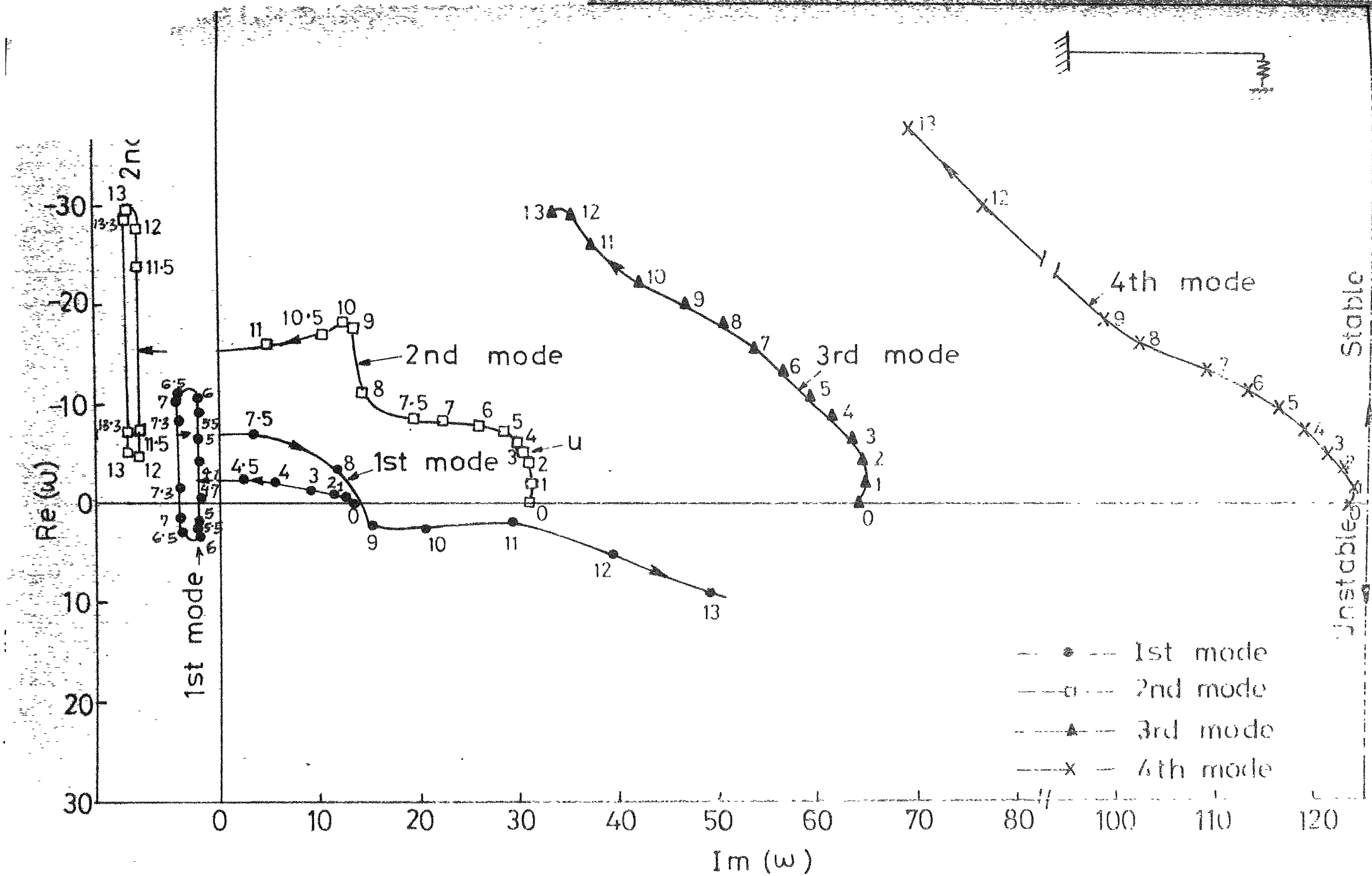


Fig. 6 Dimensionless complex frequency diagram for the system as functions of dimensionless velocity=u, $\alpha = 100{\cdot}0$, $\beta = 0{\cdot}6$, NEL = 5

$\tau = 0$ $P = 0$

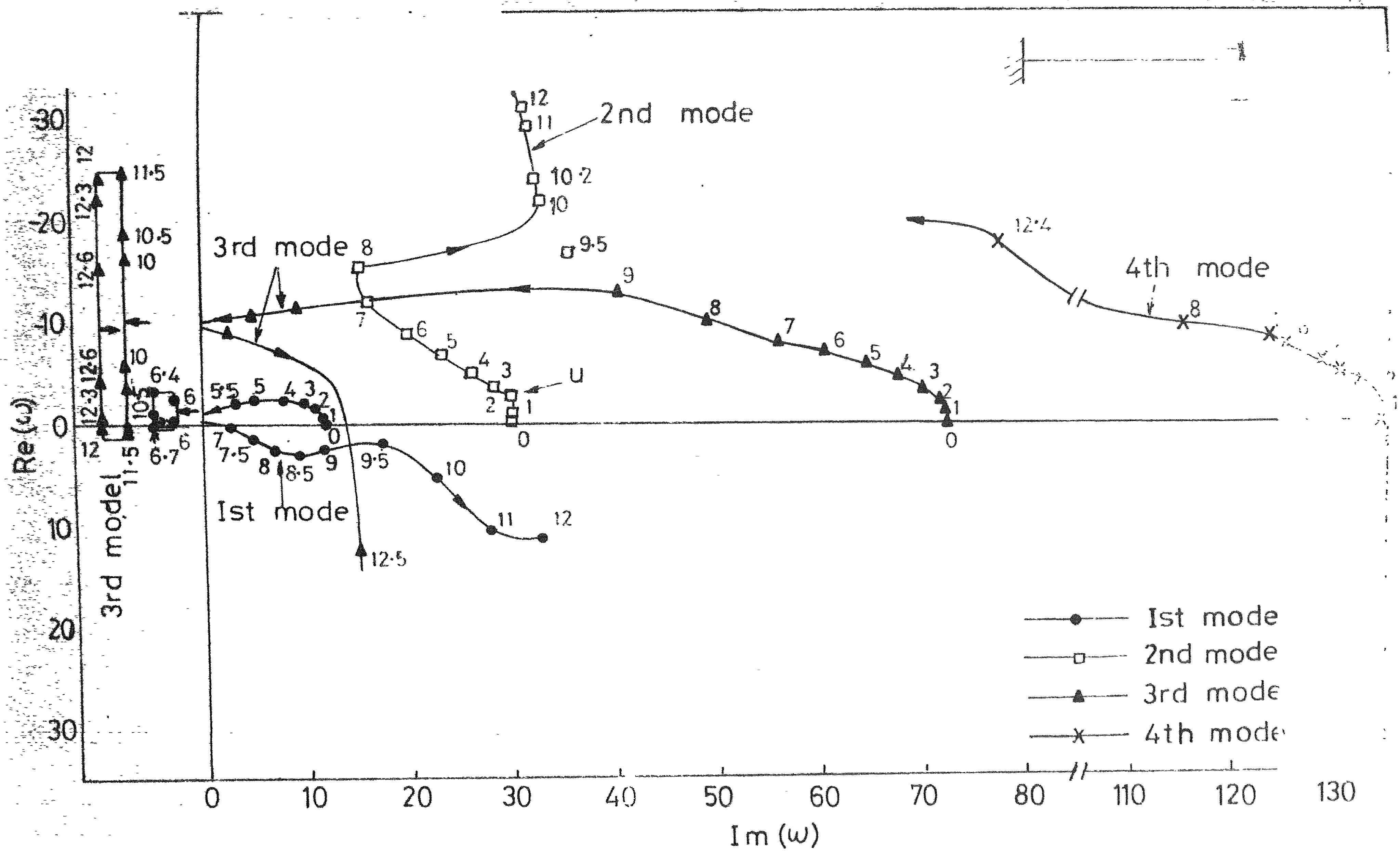


Fig. 7 Dimensionless complex frequency diagram for the system as function of dimensionless velocity $u, \alpha_d = 50 \cdot 0$, $\beta = 0 \cdot 6$, $\alpha_t = 25 \cdot 0$, NEL = 5, $\tau = 0$, $P = 0$

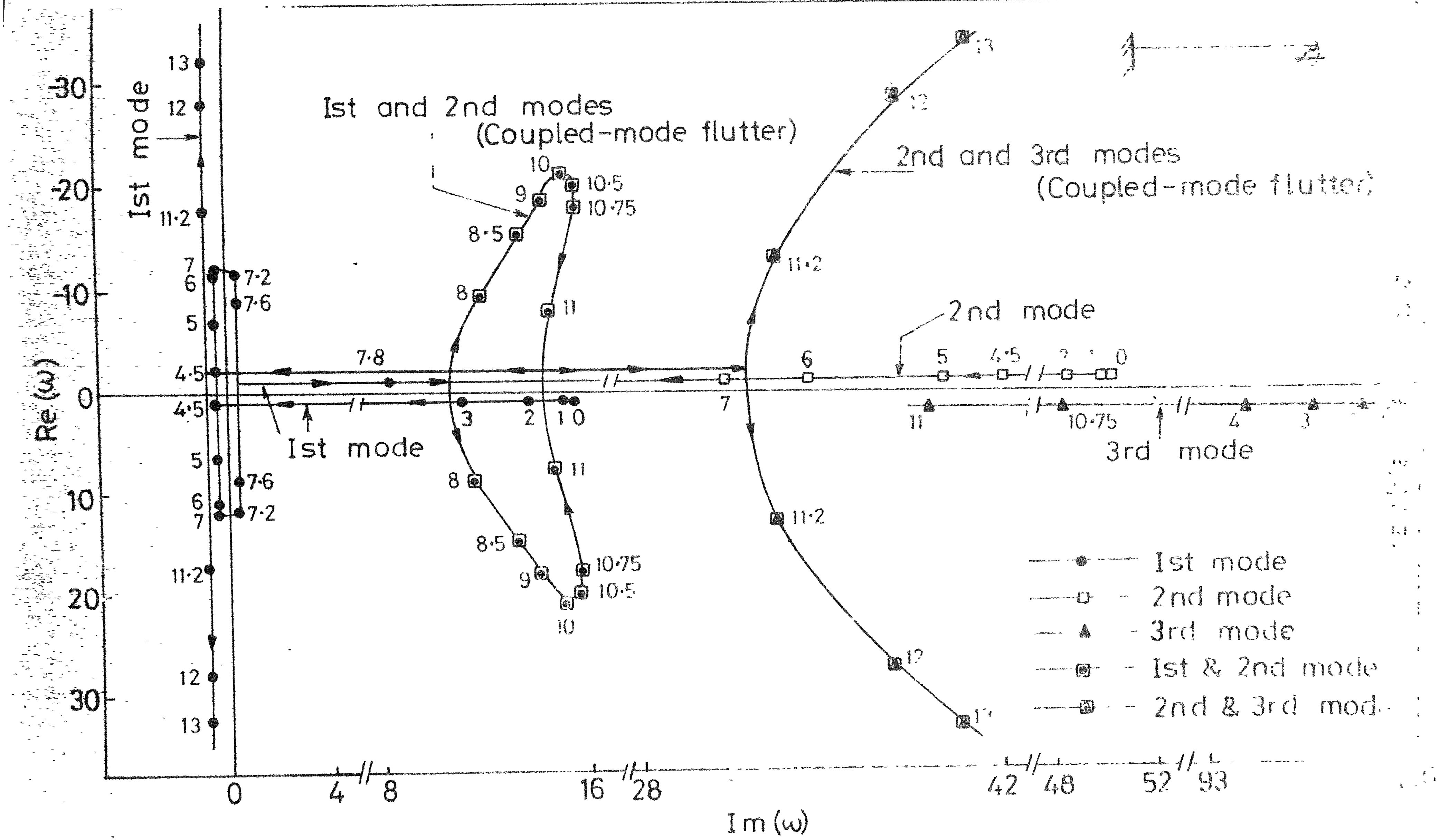


Fig. 8 Dimensionless complex frequency diagram of fixed-pinned pipe. as function of dimensionless velocity=u, β=0·5, NEL=5.

T=0, P=0

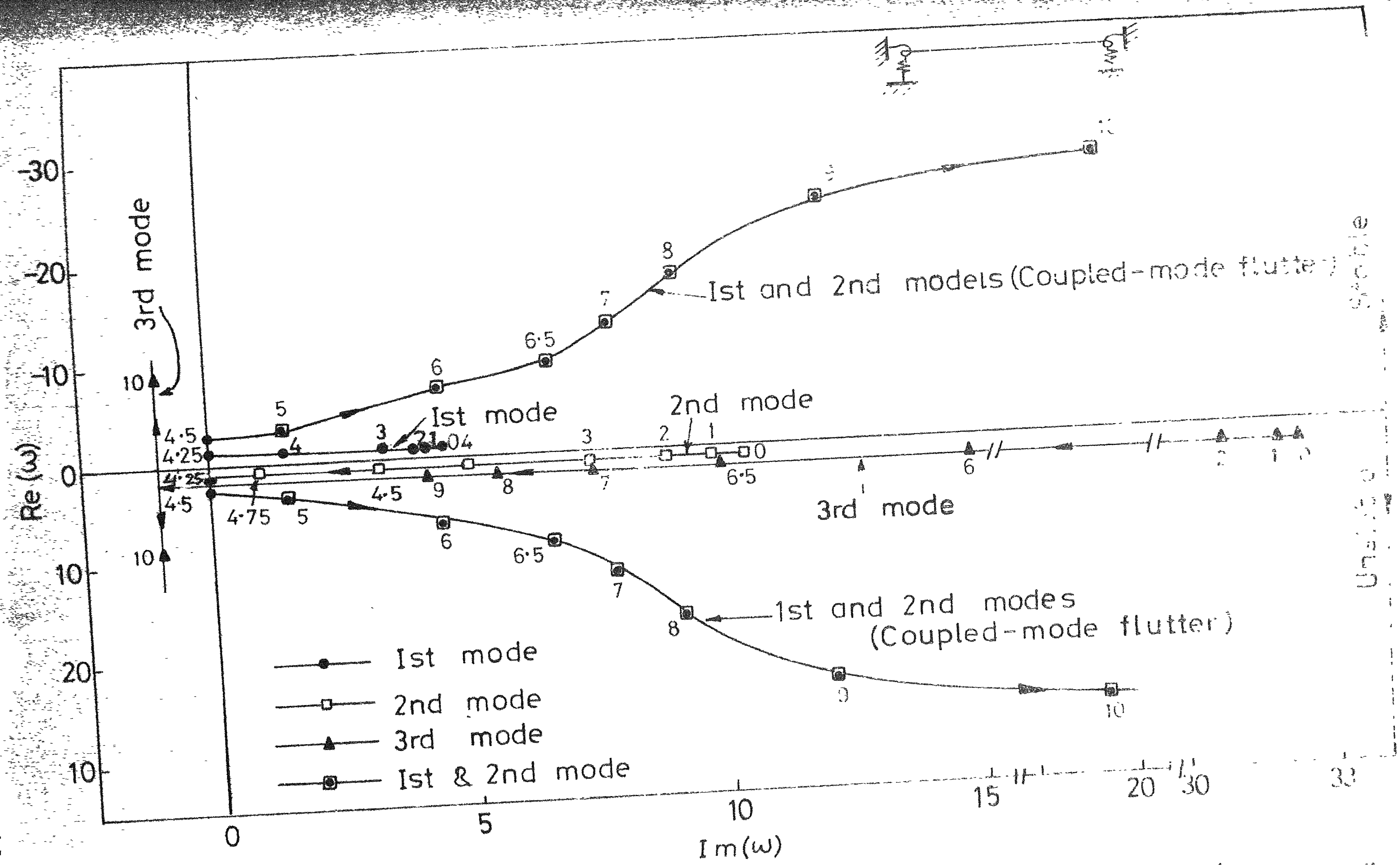


Fig. 9 Dimensionless complex frequency diagram of the system as a function of dimensionless flow velocity u, $\alpha_d = 10.0$, $\beta = 0.2$, $\alpha_1 = 5$ (Symmetric Boundaries) NEL = 5, $T = 0$, $P = 0$

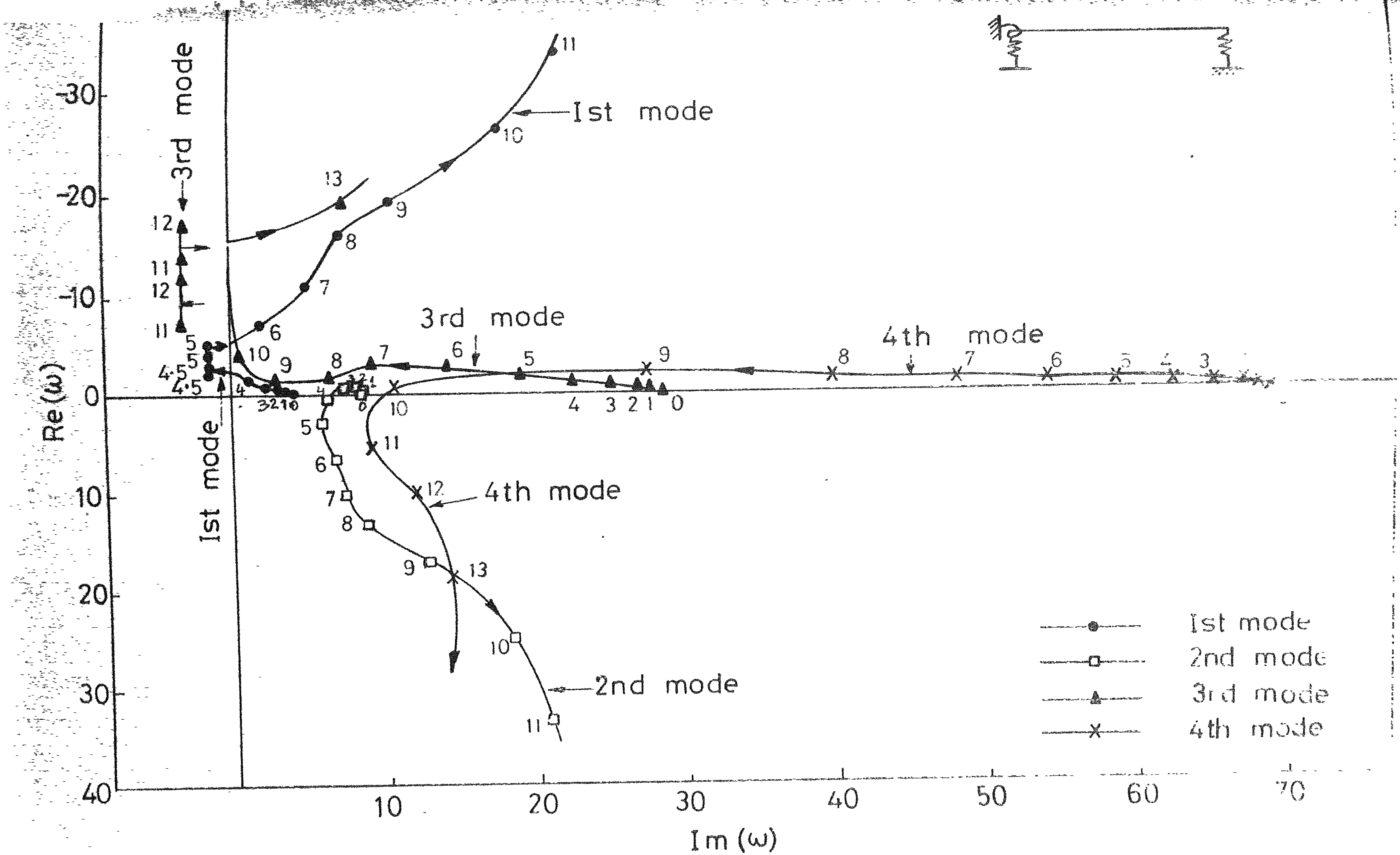


Fig. 10 Dimensionless complex frequency diagram of the system as a function of dimensionless flow velocity u, $\alpha_d = 10{\cdot}0$, $\beta = 0{\cdot}2$, $\gamma = 5{\cdot}0$ (Non-symmetric supports), $T = 0$, $P = 0$

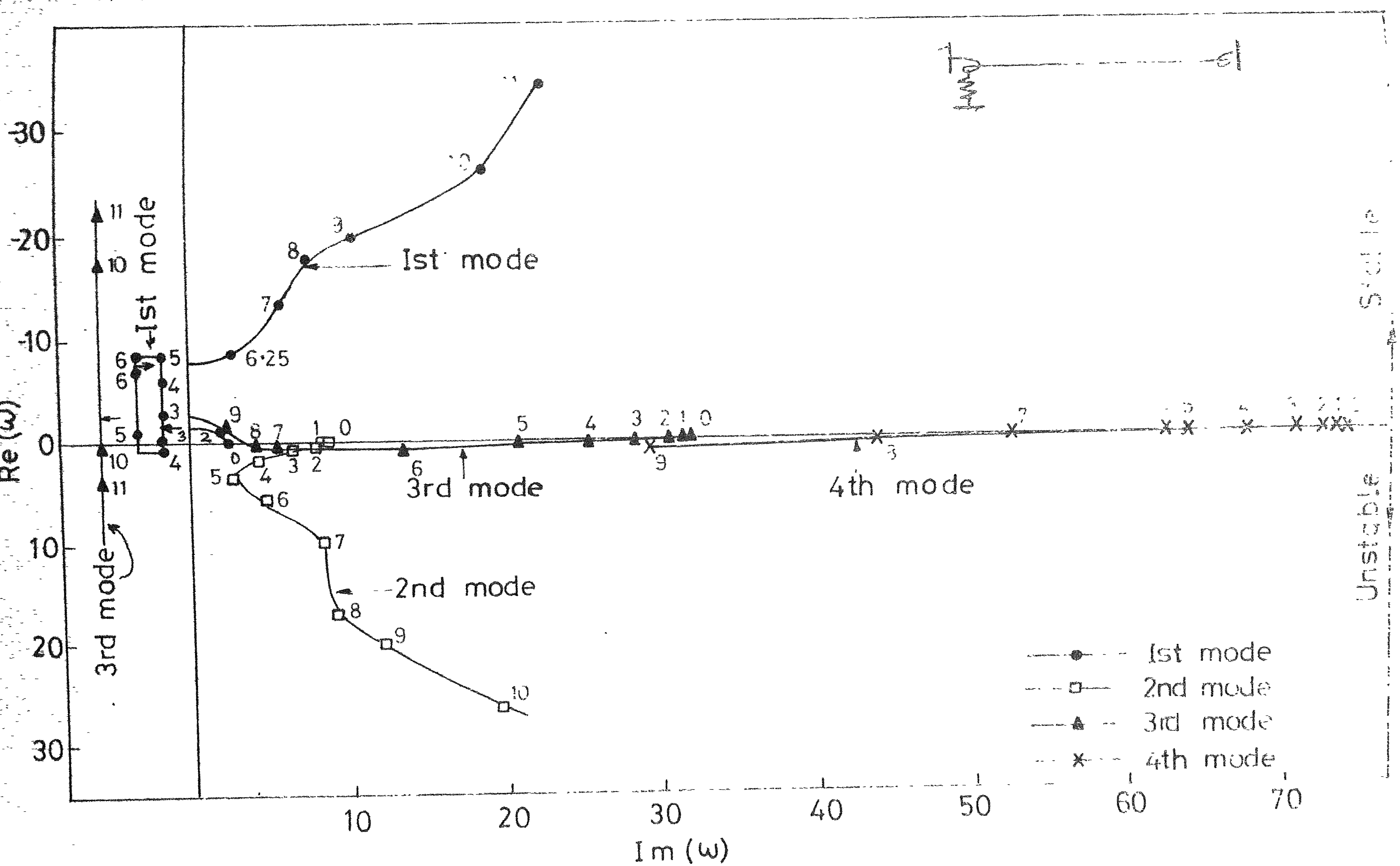


Fig. 11 Dimensionless complex frequency diagram of the system as a function of dimensionless flow velocity u, $\alpha_d = 10$, $\beta = 0{\cdot}2$, $\alpha_t = 5$ NFI = 5 (Non-symmetric supports), T = 0, P = 0

APPENDIX I

THE COMPUTER PROGRAM

The computer programme gives the complex eigenvalues of the assembled dynamical matrix. The programme was written in Fortran IV and run on an IBM 7044 computer with 32K words of core storage. The programme is quite general except that some statements will be required to be changed for different boundary conditions.

A Fortran listing of the programme is given in Appendix -II .

I.1 Description of Programme:

The MAIN programme reads and prints various data and outputs results. Other subroutines assemble the matrices and after applying boundary conditions it forms the dynamical matrix. The complex eigenvalues are calculated by calling a library subroutine

Subroutine Name	Description
ASMBL	Called by MAIN, assembles the elemental mass, damping and stiffness matrices in turn.

Subroutine Name	Description
BONDRY	Called by MAIN, applies the boundary conditions to the assembled mass, damping and stiffness matrices
DYNMAT	Called by MAIN, forms the dynamical matrix of the problem
MATINV	Called by DYNMAT, gives matrix inverse
EIGENP	Called by MAIN, determines eigenvalues and eigenvectors of a general real matrix. The binary version of this programme is stored on 1301 disk file. The name of the file is ZINDLA. For further details refer to Communication of ACM Journal, Vol 2, 1968, P. 820

LIST OF PRINCIPAL VARIABLES

FORTRAN Implementation:

Program Symbol	Definition
(MAIN)	
ALPHA	Dimensionless displacement spring constant
BEL	Length of the beam
BETA	Dimensionless mass

Program Symbol	Definition
COMNC2	Factor which adds to damping matrix (see Eqn. 2.20)
COMNK3	Displacement spring factor at left end
COMNK4	Rotational spring factor at left end
COMNK6	Displacement spring factor at right end
COMNK7	Rotational spring factor at right end
COMNK8	Factor which adds to stiffness matrix (see Eqn. 2.20)
DI	Moment of inertia of pipe
DIN	Internal diameter of pipe
DOUT	External diameter of pipe
E	Modulus of elasticity of the pipe material
ECM	Elemental damping matrix (4,4)
EKM	Elemental stiffness matrix (4,4)
EMM	Elemental mass matrix (4,4)
FACNON	Dividing factor which gives dimensionless eigenvalues
GAMA	Dimensionless rotational spring constant

Program Symbol	Definition
KB	Factor gives the No. of constraints on the boundary
NEL	Number of finite elements of the beam
NKB	Matrix size after applying boundary condition
N1	Total number of degrees of freedom
P	Fluid pressure
ROF	Density of the fluid
ROT	Density of the pipe material
STIFAX	Stiffness of the displacement spring
STIFTO	Stiffness of the rotational spring
V	Velocity of the fluid flow
VNON	Non-dimensional fluid flow velocity

SUBROUTINE ASAMBL

AM	Assembled matrix (N1, N1)
EM	Elemental matrix
N1	Total number of degrees of freedom of the system

SUBROUTINE BONDRY

IE	Row and column numbers on which boundary condition (0) is to be applied

SUBROUTINE BONDRY

K	Number of boundary conditions applied on the system
N	Number of degrees of freedom of the system

SUBROUTINE DYNMAT

DD	Dynamical matrix (N11, N11)
N1	Matrix size after applying boundary condition
N11	Size of the dynamical matrix

SUBROUTINE MATINV

A	Matrix to be inversioned and finally its inversion
DETERM	Value of the determinant

SUBROUTINE EIGENP

A	Dynamical matrix (NXN) for which eigenvalue and eigenvector calculated
EVI(I)	Imaginary part of eigenvalue
EVR(I)	Real part of eigenvalue
N	Dimension of the dynamical matrix

SUBROUTINE EIGENP

NM	Dimension of dynamical matrix given in the main program NM $\geqslant$ N
T	Number of binary digits in the mantissa of a single precision floating-point number = 27.0
VECI(J,I)	Imaginary part of eigenvector
VECR(J,I)	Real part of eigenvector

APPENDIX - III

```
$EXECUTE       DFMP
$PATCH  ZIAOLA   CCS99              010677
$OUTPUT U17=I17
$ENDFIL
$IBSYS
$IBSYS
$IBJOB
$IBFTC MAIN
C
C      ----------------------------------------------------------------------
C      FINITE ELEMENT ANALYSIS OF DYNAMIC STABILITY OF PIPE
C      CONVAYING FLUID
C      DETAILS OF THE PROGRAMME ARE GIVEN IN APPENDIX I AND II
C      CANTILEVERED PIPE WITH FREE END SUPPORTED BY A DISPLACEMENT
C      AND A ROTATIONAL SPRING
C      EMM IS THE ELEMENTAL MASS MATRIX
C      ECM IS THE ELEMENTAL DAMPING MATRIX
C      EKM IS THE ELEMENTAL STIFFNESS MATRIX
C      AM IS THE OVERALL ASSEMBLED MASS MATRIX
C      AC IS THE OVERALL ASSEMBLED DAMPING MATRIX
C      AK IS THE OVERALL ASSEMBLED STIFFNESS MATRIX
C      NEL IS THE NUMBER OF ELEMENTS
C      DM IS THE DYNAMICAL MATRIX
C      ASAMBL IS THE SUBROUTINE FOR ASSEMBLY OF MATRICES
C      V IS THE VELOCITY OF THE FLUID
C      DOUT IS THE OUT SIDE DIAMETER OF THE PIPE
C      DIN IS THE INSIDE DIAMETER OF THE PIPE
C      BEL IS THE LENGTH OF THE PIPE
C      E IS THE MODULUS OF ELASTICITY OF THE MATERIAL OF THE PIPE
C      NEL IS THE NUMBER OF ELEMENTS
C      AMF IS THE MASS OF THE FLUID PER UNIT LENGTH OF THE PIPE
C      AMT IS THE MASS OF THE PIPE PER UNIT LENGTH OF THE PIPE
C      TA IS THE TENSSION OF THE PIPE INITIALLY
C      P IS THE FLUID PRESSURE
C      AF IS THE AREA OF THE FLUID PASSAGE
C      VND IS THE NON-DIMENSIONAL VELOCITY
C      DI IS THE MOMENT OF INERTIA
C      STIFAX IS THE STIFFNESS OF THE DISPLACEMENT SPRING
C      STIFTO IS THE STIFFNESS OF THE ROTATIONAL SPRING
C      ALPHA=STIFAX*(BEL**3)/(E*DI)=NON-DIMENSIONAL DISPLACEMENT
C      SPRING FACTOR
C      BETA=AMF/(AMF+AMT)=NON-DIMENSIONAL MASS
C      GAMA=STIFTO*BEL/(E*DI)=NON-DIMENSIONAL ROTATIONAL SPRING FACTOR
C      FACNON=NON-DIMENSIONAL FACTOR WHICH WHEN DIVIDED TO EIGENVALUE
C      GIVES NON-DIMENSIONAL FREQUENCY
C      FOR CANTILEVER CASE PUT KB=2,  IE(1)=1,  IE(2)=2
C      FOR PINNED-PINNED CASE PUT KB=2,  IE(1)=1,  IE(2)=NB
```

```
C       FOR FIXED-FIXED CASE PUT KB=4, IE(1)=1, IE(2)=2
C       IE(3)=NB, IE(4)=N1
C       FOR ONE END FIXED AND OTHER END PINNED THEN PUT  KB=3,
C       IE(1)=1,  IE(2)=2,  IE(3)=NB
C       WHEN BOTH THE ENDS ARE SUPPORTED BY SPRINGS PUT KB=0
C       COMNK3=DISPLACEMENT SPRING AT LEFT END
C       COMNK4=ROTATIONAL SPRING AT THE LEFT END
C       COMNK6=DISPLACEMENT SPRING AT THE RIGHT END
C       COMNK7=ROTATIONAL SPRING AT THE RIGHT END
C       ------------------------------------------------------------------
        DIMENSION EMM(4,4),ECM(4,4),EKM1(4,4),EKM2(4,4),EKM(4,4),AM(25,25)
        DIMENSION AC(25,25),AK(25,25),DD(50,50)
        DIMENSION EVR(50),EVI(50),VECR(50,50),VECI(50,50),INDIC(50)
        DIMENSION EVRN(50),EVIN(50)
        DIMENSION IE(10)
        COMMON/DYN1/AM,AC
        COMMON/DYN2/DD
        COMMON/DYN3/FACTOR
        COMMON/DYN4/NEL,VNON
        COMMON/DYN5/AK
        DATA DOUT,DIN,BEL,ROT/0.024,0.02,2.0,7800.0/
        DATA E,TA,P/0.204E+12,0.0,0.0/
        DATA ALPHA,BETA,GAMA/10.0,0.2,5.0/
        KB=2
        PRINT 15,DOUT,DIN,BEL
   15   FORMAT(/,5X,*DOUT=*,E15.7,*METERS*,2X,*DIN=*,E15.7,*METERS*,2X,
       1*BEL=*,E15.7,*METERS*)
        PRINT 35,TA,P
   35   FORMAT(/,5X,*TA=*,E15.7,*NEWTONS*,2X,*P=*,E15.7,*N/SQ.M*)
        PI=4.0*ATAN(1.)
        CALL FLUN(20000)
        CALL FLOV(20000)
        ROF=(BETA/(1.0-BETA))*ROT*((DOUT+DIN)*(DOUT-DIN)/(DIN**2))
        PRINT 25,ROT,ROF,E
   25   FORMAT(/,5X,*ROT=*,E15.7,*KG/CU.M*,2X,*ROF=*,E15.7,*KG/CU.M*,
       22X,*E=*,E15.7,*N/SQ.M*)
        AF=PI*(DIN**2)/4.0
        AT=PI*(DOUT+DIN)*(DOUT-DIN)/4.0
        AMT=PI*ROT*(DOUT+DIN)*(DOUT-DIN)/4.0
        AMF=ROF*AF
        PRINT 145,AMT,AMF
  145   FORMAT(/,5X,*AMT=*,E15.7,5X,*AMF=*,E15.7)
        DI=PI*((DOUT**2)+(DIN**2))*(DOUT+DIN)*(DOUT-DIN)/64.0
        SQT=SQRT(E*DI/AMF)/BEL
        FACNON=SQRT((E*DI)/((AMF+AMT)*(BEL**4)))
        STIFAX=ALPHA*E*DI/(BEL**3)
        STIFTO=GAMA*E*DI/BEL
        PRINT 105,DI,FACNON,STIFAX,STIFTO
  105   FORMAT(/,5X,*DI=*,E15.7,2X,*FACNON=*,E15.7,2X,
       3*STIFAX=*,E15.7,2X,*STIFTO=*,E15.7)
```

```
      PRINT 175,ALPHA,BETA,GAMA
  175 FORMAT(/,5X,*ALPHA=*,E15.7,5X,*BETA=*,E15.7,5X,*GAMA=*,E15.7)
      WRITE(6,86)
   86 FORMAT(/,2X,125(1H*))
      NEL=6
      DO 100 NNEL=1,6
      EN=NEL
      EL =BEL/EN
      PRINT 135,NEL
  135 FORMAT(/,2X,*NEL=*,I3)
      VNON=0.0
      DO 30 NV=1,15
      V=VNON*SQT
      COMNM=(AMF+AMT)*EL/420.0
      COMNC=AMF*V/30.0
      COMNC2=2.0*AMF*V
      COMNK1=E*DI/(EL**3)
      COMNK2=(TA-(P*AF)-(AMF*(V**2)))/(30.0*EL)
      COMNK3=STIFAX
      COMNK4=STIFTC
      COMNK5=AMF*(V**2)*EL
      COMNK6=COMNK3
      COMNK7=COMNK4
      COMNK8=(AMF*(V**2))-TA+(P*AF)
      PRINT 115,VNON,V
  115 FORMAT(/,5X,*VNON=*,E15.7,2X,*V=*,E15.7,*METERS/SEC*)
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 27
      PRINT 125,COMNM,COMNC,COMNK1,COMNK2
  125 FORMAT(/,5X,*COMNM=*,E15.7,5X,*COMNC=*,E15.7,5X,*COMNK1=*,E15.7,
     25X,*COMNK2=*,E15.7)
   27 CONTINUE
      N1=2*(NEL+1)
C     ************************************************************************
C     FORM THE ELEMENTAL MASS MATRIX
C     ************************************************************************
      EMM(1,1)=COMNM*156.0
      EMM(1,2)=COMNM*22.0*EL
      EMM(1,3)=COMNM*54.0
      EMM(1,4)=-COMNM*13.0*EL
      EMM(2,1)=EMM(1,2)
      EMM(2,2)=COMNM*4.0*(EL**2)
      EMM(2,3)=-EMM(1,4)
      EMM(2,4)=-COMNM*3.0*(EL**2)
      EMM(3,1)=EMM(1,3)
      EMM(3,2)=EMM(2,3)
      EMM(3,3)=EMM(1,1)
      EMM(3,4)=-EMM(1,2)
      EMM(4,1)=EMM(1,4)
      EMM(4,2)=EMM(2,4)
      EMM(4,3)=EMM(3,4)
```

```
      EMM(4,4)=EMM(2,2)
C     ------------------------------------------------------------
C     FORM THE OVERALL MASS MATRIX
      CALL ASAMBL(NEL,EMM,N1,AM)
C     ************************************************************
C     FORM THE ELEMENTAL DAMPING MATRIX
C     ************************************************************
      ECM(1,1)=COMNC*30.0
      ECM(1,2)=COMNC*6.0*EL
      ECM(1,3)=ECM(1,1)
      ECM(1,4)=-ECM(1,2)
      ECM(2,1)=-ECM(1,2)
      ECM(2,2)=C.0
      ECM(2,3)=-ECM(2,1)
      ECM(2,4)=-COMNC*(EL**2)
      ECM(3,1)=-ECM(1,3)
      ECM(3,2)=-ECM(2,3)
      ECM(3,3)=-ECM(1,1)
      ECM(3,4)=-ECM(3,2)
      ECM(4,1)=-ECM(1,4)
      ECM(4,2)=-ECM(2,4)
      ECM(4,3)=-ECM(3,4)
      ECM(4,4)=0.0
C     ------------------------------------------------------------
C     FORM THE OVERALL DAMPING MATRIX
      CALL ASAMBL(NEL,ECM,N1,AC)
      AC(1,1)=AC(1,1)-COMNC2
      AC(NB,NB)=AC(NB,NB)+COMNC2
C     ************************************************************
C     FORM THE ELEMENTAL STIFFNESS MATRIX
      EKM1(1,1)=COMNK1*12.0
      EKM1(1,2)=COMNK1*6.0*EL
      EKM1(1,3)=-EKM1(1,1)
      EKM1(1,4)=EKM1(1,2)
      EKM1(2,1)=EKM1(1,2)
      EKM1(2,2)=COMNK1*4.0*(EL**2)
      EKM1(2,3)=-EKM1(2,1)
      EKM1(2,4)=COMNK1*2.0*(EL**2)
      EKM1(3,1)=EKM1(1,3)
      EKM1(3,2)=EKM1(2,3)
      EKM1(3,3)=EKM1(1,1)
      EKM1(3,4)=EKM1(2,3)
      EKM1(4,1)=EKM1(1,4)
      EKM1(4,2)=EKM1(2,4)
      EKM1(4,3)=EKM1(3,4)
      EKM1(4,4)=EKM1(2,2)
C     ------------------------------------------------------------
      EKM2(1,1)=COMNK2*36.0
      EKM2(1,2)=COMNK2*3.0*EL
      EKM2(1,3)=-EKM2(1,1)
```

```
      EKM2(1,4)=EKM2(1,2)
      EKM2(2,1)=EKM2(1,2)
      EKM2(2,2)=COMNK2*4.0*(EL**2)
      EKM2(2,3)=-EKM2(2,1)
      EKM2(2,4)=-COMNK2*(EL**2)
      EKM2(3,1)=EKM2(1,3)
      EKM2(3,2)=EKM2(2,3)
      EKM2(3,3)=EKM2(1,1)
      EKM2(3,4)=EKM2(2,3)
      EKM2(4,1)=EKM2(1,4)
      EKM2(4,2)=EKM2(2,4)
      EKM2(4,3)=EKM2(3,4)
      EKM2(4,4)=EKM2(2,2)
C     ------------------------------------------------------------------
      DO 10 I=1,4
      DO 10 J=1,4
      EKM(I,J)=EKM1(I,J)+EKM2(I,J)
   10 CONTINUE
C     ------------------------------------------------------------------
C     FORM THE OVERALL STIFFNESS MATRIX
      CALL ASAMBL(NEL,EKM,N1,AK)
C     ------------------------------------------------------------------
      AK(1,1)=AK(1,1)+COMNK3
      AK(2,2)=AK(2,2)+COMNK4
      AK(NB,NB)=AK(NB,NB)+COMNK6
      AK(N1,N1)=AK(N1,N1)+COMNK7
      AK(1,2)=AK(1,2)-COMNK5
      AK(NB,N1)=AK(NB,N1)+COMNK8
C     ------------------------------------------------------------------
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 37
      WRITE(6,16)
   16 FORMAT(/,12X,*THE OVERALL MASS MATRIX*)
      PRINT 45,((AM(I,J),J=1,N1),I=1,N1)
   45 FORMAT(2X,6E15.7)
      WRITE(6,26)
   26 FORMAT(/,12X,*THE OVERALL DAMPING MATRIX*)
      PRINT 55,((AC(I,J),J=1,N1),I=1,N1)
   55 FORMAT(2X,6E15.7)
      WRITE(6,36)
   36 FORMAT(/,12X,*THE OVERALL STIFFNESS MATRIX*)
      PRINT 65,((AK(I,J),J=1,N1),I=1,N1)
   65 FORMAT(2X,6E15.7)
   37 CONTINUE
C     ------------------------------------------------------------------
C     APPLYING THE BOUNDARY CONDITIONS
C     KB IS DEFINED EARLIER FOR DIFFERENT BOUNDARY CONDITIONS
      N1=2*(NEL+1)
      NB=N1-1
      NKB=N1-KB
      IF(KB.EQ.0) GO TO 90
```

```
      I (1)=1
      I (2)=2
      DO 40 I=1,N1
      DO 40 J=1,N1
      DD(I,J)=AM(I,J)
   40 CONTINUE
      CALL BONDRY(IE,KB,N1)
      DO 70 I=1,NKB
      DO 70 J=1,NKB
      AM(I,J)=DD(I,J)
   70 CONTINUE
      DO 50 I=1,N1
      DO 50 J=1,N1
      DD(I,J)=AC(I,J)
   50 CONTINUE
      CALL BONDRY(IE,KB,N1)
      DO 80 I=1,NKB
      DO 80 J=1,NKB
      AC(I,J)=DD(I,J)
   80 CONTINUE
      DO 60 I=1,N1
      DO 60 J=1,N1
      DD(I,J)=AK(I,J)
   60 CONTINUE
      CALL BONDRY(IE,KB,N1)
      DO 90 I=1,NKB
      DO 90 J=1,NKB
      AK(I,J)=DD(I,J)
   90 CONTINUE
C     ------------------------------------------------------------------------
      IF(NEL.NE.2.OR.VMON.NE.1.0) GO TO 47
      WRITE(6,46)
   46 FORMAT(/,12X,*MASS MATRIX AFTER BOUNDARY CONDITIONS*)
      PRINT 75,((AM(I,J),J=1,NKB),I=1,NKB)
   75 FORMAT(2X,2E15.7)
      WRITE(6,56)
   56 FORMAT(/,12X,*DAMPING MATRIX AFTER BOUNDARY CONDITION*)
      PRINT 85,((AC(I,J),J=1,NKB),I=1,NKB)
   85 FORMAT(2X,2E15.7)
      WRITE(6,66)
   66 FORMAT(/,12X,*STIFFNESS MATRIX AFTER BOUNDARY CONDITION*)
      PRINT 95,((AK(I,J),J=1,NKB),I=1,NKB)
   95 FORMAT(2X,2E15.7)
   47 CONTINUE
C     FORM THE DYNAMICAL MATRIX
C     ------------------------------------------------------------------------
      CALL DYNMAT(NKB,N11)
      NM=50
      T=27.0
      CALL EIGENP(N11,NM,DD,T,EVR,EVI,VECR,VECI,INDIC)
```

```
      DO 110 I=1,N11
      DENOM=((EVR(I)**2)+(EVI(I)**2))
      EVR(I)=EVR(I)/DENOM
      EVI(I)=-EVI(I)/DENOM
      EVRN(I)=EVR(I)/FACNON
      EVIN(I)=EVI(I)/FACNON
  110 CONTINUE
      WRITE(6,76)
   76 FORMAT(/,17X,*EVR*,17X,*EVI*,12X,*II*,10X,*EVRN*,17X,*EVIN*)
      PRINT 155,(EVR(I),EVI(I),EVRN(I),EVIN(I),I=1,10)
  155 FORMAT(11X,E16.8,5X,E16.8,4X,*II*,4X,E16.8,5X,E16.8)
      PRINT 165,(INDIC(I),I=1,N11)
  165 FORMAT(5X,15I5)
      WRITE(6,96)
   96 FORMAT(/,2X,125(1H-))
      VNON=VNON+1.0
   30 CONTINUE
      NEL=NEL+1
      WRITE(6,106)
  106 FORMAT(/,2X,125(1H*))
  100 CONTINUE
      STOP
      END
$IBFTC ASAMBL
      SUBROUTINE ASAMBL(NEL,EM,N1,AM)
C     *********************************************************************
C     SUBROUTINE FOR THE ASSEMBLY OF MATRICES
C     *********************************************************************
      DIMENSION EM(4,4),AM(25,25)
      N=NEL
      N1=2*(N+1)
      DO 10 I=1,N1
      DO 10 J=1,N1
      AM(I,J)=0.0
   10 CONTINUE
      DO 20 INEL=1,N
      N2=(2*INEL)-1
      N3=(2*INEL)+2
      N4=(2*INEL)-2
      DO 20 I=N2,N3
      DO 20 J=N2,N3
      N5=I-N4
      N6=J-N4
      AM(I,J)=AM(I,J)+EM(N5,N6)
   20 CONTINUE
      RETURN
      END
$IBFTC DYNMAT
      SUBROUTINE DYNMAT(N1,N11)
C     *********************************************************************
```

```
C     SUBROUTINE FOR THE ASSEMBLY OF DYNAMICAL MATRIX
C     ***************************************************************
C     KIB IS THE REDUCED SIZE OF THE MATRIX AFTER BOUNDARY CONDITION
C     SCALING THE STIFFNESS MATRIX AK
C     FTR IS THE BIGGEST ELEMENT OF THE STIFFNESS MATRIX AK
      DIMENSION AM(25,25),AC(25,25),AK(25,25),DD(50,50),AMC(25)
      COMMON/DYN1/AM,AC
      COMMON/DYN2/DD
      COMMON/DYN3/FACTOR
      COMMON/DYN4/NEL,VNON
      COMMON/DYN5/AK
      FTR=ABS(AK(1,1))
      DO 60 I=1,N1
      DO 60 J=1,N1
      IF(ABS(AK(I,J)).GT.FTR) FTR=ABS(AK(I,J))
   60 CONTINUE
      DO 70 I=1,N1
      DO 70 J=1,N1
      AK(I,J)=AK(I,J)/FTR
   70 CONTINUE
C     CALLING THE MATRIX INVERSE
      CALL MATINV(N1,DETERM)
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 37
      WRITE(6,16)
   16 FORMAT(/,12X,*INVERSE OF THE SCALED MATRIX AK*)
      PRINT 25,((AK(I,J),J=1,N1),I=1,N1)
   25 FORMAT(2X,2E15.7)
   37 CONTINUE
C     MULTIPLY AK INVERSE AND AM AND STORE IN AM
C     ---------------------------------------------------------------
      DO 20 J=1,N1
      DO 30 I=1,N1
      SUM=0.0
      DO 40 K=1,N1
      SUM=SUM+AK(I,K)*AM(K,J)
   40 CONTINUE
      AMC(I)=SUM/FTR
   30 CONTINUE
      DO 50 L=1,N1
      AM(L,J)=AMC(L)
   50 CONTINUE
   20 CONTINUE
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 47
      WRITE(6,26)
   26 FORMAT(/,12X,*PRODUCT OF AK INVERSE AND AM*)
      PRINT 35,((AM(I,J),J=1,N1),I=1,N1)
   35 FORMAT(2X,2E15.7)
   47 CONTINUE
C     MULTIPLY INVERSE OF AK AND AC AND STORE IN AC
C     ---------------------------------------------------------------
```

```
      DO 110 J=1,N1
      DO 120 I=1,N1
      SUM=0.0
      DO 130 K=1,N1
      SUM=SUM+AK(I,K)*AC(K,J)
  130 CONTINUE
      AMC(I)=SUM/FTR
  120 CONTINUE
      DO 140 L=1,N1
      AC(L,J)=AMC(L)
  140 CONTINUE
  110 CONTINUE
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 57
      WRITE(6,36)
   36 FORMAT(/,12X,*PRODUCT OF AK INVERSE AND AC*)
      PRINT 45,((AC(I,J),J=1,N1),I=1,N1)
   45 FORMAT(2X,2E15.7)
   57 CONTINUE
C     ------------------------------------------------------------------
C     FORM THE DYNAMICAL MATRIX
C     ------------------------------------------------------------------
      N11=2*N1
      DO 80 I=1,N11
      DO 80 J=1,N11
      DD(I,J)=0.0
   80 CONTINUE
      DO 90 I=1,N1
      DO 90 J=1,N1
      NN1=N1+I
      NN2=N1+J
      DD(NN1,J)=-AM(I,J)
      DD(I,NN1)=1.0
      DD(NN1,NN2)=-AC(I,J)
   90 CONTINUE
C     FACTOR IS THE FACTOR TAKEN COMMON FROM THE MATRIX
      FACTOR=1.0
      DO 100 I=1,N11
      DO 100 J=1,N11
      DD(I,J)=DD(I,J)/FACTOR
  100 CONTINUE
      IF(NEL.NE.2.OR.VNON.NE.1.0) GO TO 67
      WRITE(6,46)
   46 FORMAT(/,12X,*DYNAMICAL MATRIX*)
      PRINT 55,((DD(I,J),J=1,N11),I=1,N11)
   55 FORMAT(2X,4E15.7)
   67 CONTINUE
      RETURN
      END
$IBFTC MATINV
      SUBROUTINE MATINV(N,DETERM)
```

```
C       ****************************************************************************
C       SUBROUTINE FOR THE MATRIX INVERTION
C       ****************************************************************************
C       ROUTINE FOR MATRIX INVERSION AND SOLUTION OF SIMULTANEOUS EQNS.
C       A IS THE COEFFICIENT MATRIX OF SIZE N
C       B IS THE VECTOR OF THE SIZE N
C       M IS NUMBER OF CONSTENT VECTORS
C       IF M IS SET TO ZERO ,ONLY INVERSE IS COMPUTED
C       DETERM IS THE VALUE OF DETERMENET RETURNED BY THE ROUTINE
C       ----------------------------------------------------------------------------
        DIMENSION A(25,25),IPIVOT(40),INDEX(40,2)
        COMMON/DYN5/A
        EQUIVALENCE (IROW,JROW),(ICOLUM,JCOLUM),(AMAX,T,SWAP)
C         INITIALIZATION
   10   DETERM=1.0
   15   DO 20 J=1,N
   20   IPIVOT(J)=0
C       SEARCH FOR PIVOT ELEMENT
   30   DO 550 I=1,N
   40   AMAX=0.0
   45   DO 105 J=1,N
   50   IF(IPIVOT(J)-1) 60,105,60
   60   DO 100 K=1,N
   70   IF(IPIVOT(K)-1) 80,100,740
   80   IF(AMAX-ABS(A(J,K))) 85,100,100
   85   IROW=J
   90   ICOLUM=K
   95   AMAX=ABS(A(J,K))
  100   CONTINUE
  105   CONTINUE
  110   IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C       INTERCHANGE THE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
  130   IF(IROW-ICOLUM) 140,260,140
  140   DETERM=-DETERM
  150   DO 200 L=1,N
  160   SWAP=A(IROW,L)
  170   A(IROW,L)=A(ICOLUM,L)
  200   A(ICOLUM,L)=SWAP
  260   INDEX(I,1)=IROW
  270   INDEX(I,2)=ICOLUM
C       DIVIDE PIVOT ROW BY PIVOT ELEMENT
  310   PIVOT=A(ICOLUM,ICOLUM)
  320   DETERM=DETERM*PIVOT
  330   A(ICOLUM,ICOLUM)=1.0
  340   DO 350 L=1,N
  350   A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
C       REDUCE NON PIVOT ROWS
  360   DO 550 L1=1,N
  390   IF(L1-ICOLUM) 400,550,400
  400   T=A(L1,ICOLUM)
```

```
   420 A(L1,ICOLUM)=0.0
   430 DO 450 L=1,N
   450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
   550 C NTINUE
C       INTERCHANG COLUMNS
   600 DO 710 I=1,N
   610 L=N+1-I
   620 IF(INDEX(L,1)-INDEX(L,2)) 630,710,630
   630 JROW=INDEX(L,1)
   640 JCOLUM=INDEX(L,2)
   650 DO 705 K=1,N
   660 SWAP=A(K,JROW)
   670 A(K,JROW)=A(K,JCOLUM)
   700 A(K,JCOLUM)=SWAP
   705 CONTINUE
   710 CONTINUE
       DO 11 K=1,N
       IF(IPIVOT(K).NE.1) GO TO 12
   11  CONTINUE
       RETURN
   12  PRINT 991
   991 FORMAT(/30X*MATRIX IS SINGULAR*/)
   740 RETURN
       END
$IBFTC BONDRY
       SUBROUTINE BONDRY(IE,K,N)
C      ****************************************************************************
C      SUBROUTINE FOR THE BOUNDARY CONDITION
C      ****************************************************************************
C      IE(I) IS THE ROW OR CCLUMN NUMBER FOR WHICH THE
C      BOUNDARY CONDITION IS APPLIED AND IS ZERO
C      K IS THE TOTAL NO. OF BOUNDARY CONDITIONS APPLIED
C      N IS THE SIZE OF THE CRIGINAL MATRIX BEFOFE APPLYING THE
C      BOUNDARY CONDIIIONS
C      DD IS THE MATRIX
C      ----------------------------------------------------------------------------
       DIMENSION IE(4),DD(50,50)
       COMMON/DYN2/DD
       KR=0
       DO 10 I=1,N
       DO 20 L=1,K
       IF(I.EQ.IE(L)) GO TO 10
   20  CONTINUE
       KR=KR+1
       KC=0
       DO 40 J=1,N
       DO 30 L=1,K
       IF(J.EQ.IE(L)) GO TO 40
   30  CONTINUE
       KC=KC+1
```

```
      CD(KR,KC)=CD(I,J)
   40 CONTINUE
    1 CONTINUE
      RETURN
      END
$IEDIT             U17,SRCH,REWIND
$IBLDR EIGINP
$IBLDR SCALE
$IBLDR HESQR
$IBLDR REALVE
$IBLDR COMPVE
$ENTRY
```